Functional methods of QFT

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Lecture 1. Canonical formalism of dynamical systems and local gauge invariance

- Canonical formalism and canonical quantization.
- From quantum mechanics (QM) to quantum field theory (QFT). Lorentz invariance, ghosts and gauge invariance.
- Relativistic particle.
- Electromagnetic and Yang-Mills (YM) fields.

1.1 Canonical formalism and canonical quantization

As is well known from quantum mechanics physical phenomena have quantum nature, and the same is expected to be true for more complicated than mechanical physical systems described by fields. So quantum field theory is the extension of known principles of canonical quantization to field systems with infinite number of degrees of freedom. So let us start describing this extension by first giving the most general formulation of canonical quantization for a generic dynamical system with a finite number of degrees of freedom — generalized coordinates q^i, $i = 1, 2, ...n$. Let the action of this system with the Lagrangian $L(q, \dot{q})$ be

$$S[q] = \int dt\, L(q, \dot{q}), \quad \dot{q}^i \equiv \frac{dq^i}{dt}. \tag{1.1}$$

Classically the evolution of these coordinates $q^i(t)$ is determined by Euler-Lagrange equations of motion which can be converted to the form of first order differential equations in time by the transition to the canonical formalism. This begins with the introduction of canonical momenta p_i conjugated to coordinates

$$p_i = \frac{\partial L}{\partial \dot{q}^i}, \tag{1.2}$$

which are in one-to-one correspondence with the velocities $\dot{q}^i = \det q^i(q, p)$ as functions of q and p provided the invertibility of the Hessian matrix $\partial^2 L/\partial\dot{q}^i\partial\dot{q}^j$ of $L(q, \dot{q})$ with respect to velocities,

$$\det \frac{\partial^2 L}{\partial\dot{q}^i\partial\dot{q}^j} \neq 0 \Rightarrow \dot{q}^i = \dot{q}^i(q, p). \tag{1.3}$$

Then the Legendre transform to the Hamiltonian $H(q, p)$ – the function on the phase space of coordinates and momenta,

$$L(q, \dot{q}) \to H(q, p) = \big[\, p_i \dot{q}^i - L(q, \dot{q}) \,\big]_{\dot{q} = \dot{q}(q,p)}, \tag{1.4}$$

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allows one to rewrite the *Lagrangian* action (1.1) in the canonical form

$$S[q,p]=\int dt\,\big(p_i\dot q^i-H(q,p)\big). \tag{1.5}$$

Its variation with respect to phase space coordinates and momenta treated as independent variables yields the Hamiltonian equations of motion,

$$\dot q^i=\frac{\partial H}{\partial p_i}=\{q^i,H\},\quad \dot p_i=-\frac{\partial H}{\partial q^i}=\{p_i,H\}, \tag{1.6}$$

in terms of the Poisson bracket defined for any pair of phase space functions $A=A(q,p)$ and $B(q,p)$

$$\{A,B\}=\frac{\partial A}{\partial q^i}\frac{\partial B}{\partial p_i}-\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q^i}. \tag{1.7}$$

Canonical quantization of such a system then consists in promoting the phase space variables and their Hamiltonian to the level of operators $\hat q^i$, $\hat p_i$ and $\hat H$, which act in the Hilbert space of physical states $|\psi(t)\rangle$ and satisfy instead of the Poisson bracket relations the canonical commutation relations

$$q,p\mapsto\hat q,\hat p,\quad H\mapsto\hat H,\quad \{q^i,p_j\}=\delta^i_j\mapsto[\,\hat q^i,\hat p_j\,]=i\hbar\delta^i_j. \tag{1.8}$$

The physical state is supposed to evolve in time via the Schroedinger equation with the quantum Hamiltonian $\hat H$,

$$i\hbar\frac{\partial}{\partial t}\,|\psi(t)\rangle=\hat H\,|\psi(t)\rangle\,. \tag{1.9}$$

Within such most general setup quantization consists in the solution of two major problems – the calculation of the probability of transition between the initial state $|\psi_1(t_1)\,\rangle$ prescribed at some initial moment of time t_1 and the final state $|\psi_2(t_2)\,\rangle$ at t_2 or the calculation of the expectation value of some physical observable $\hat{\mathcal O}$ in the evolving quantum state as a function of time

$$\begin{cases}|\,\langle\,\psi_2(t_2)\,|\,\psi_1(t_1)\,\rangle\,|^2=P_{1\to2}\\[2ex] \langle\,\psi(t)\,|\,\hat{\mathcal O}\,|\,\psi(t)\,\rangle=\langle\,\mathcal O\,\rangle\,(t).\end{cases} \tag{1.10}$$

1.2 From QM to QFT. Lorentz invariance, ghosts and gauge invariance

To simplify the formalism we will in what follows work in the universal system of units and also use the $(-+++)$-signature of the flat spacetime Lorentzian metric $\eta_{\mu\nu}$,

$$\hbar=c=1,\quad \eta_{\mu\nu}=\mathrm{diag}(-1,1,1,1), \tag{1.11}$$

where Greek indices will be the labels of spacetime coordinates $x=x^\mu$, $\mu=0,1,2,3$, $x^0=t$, while spatial coordinates $\mathbf x=x^i$ will be basically labeled by the letters from the second half of Latin alphabet. The transition from QM with finite number of degrees of freedom to QFT of a scalar $\phi(x)$, spinor $\psi^A(x)$ (A is the spinor index), vector $A_\mu(x)$, metric $g_{\mu\nu}(x)$, etc. fields

$$q^i(t)\mapsto\phi(t,\mathbf x);\ \psi^A(t,\mathbf x),\ A_\mu(t,\mathbf x),\ g_{\mu\nu}(t,\mathbf x),\ ...\,, \tag{1.12}$$

implies that the index i of $q^i(t)$ acquires an infinite continuous range, $i\mapsto\mathbf x$, $(\mathbf x,A)$, $(\mathbf x,\mu)$, $(\mathbf x,\mu\nu)$, ... associated with the continuum of spatial points.

For Lorentz-invariant theories, which we will basically consider, all these fields belong to the representation of $O(3,1)$ Lorentz group preserving the Lorentzian metric $\eta_{\mu\nu}$. Starting with spin one vector field this immediately leads to the following problem. For the sake of Lorentz invariance the vector index of $A_\alpha=(A_0,\boldsymbol A)$ in the Lagrangian should be obviously contracted with the aid of the contravariant metric $\eta^{\alpha\beta}$. For the kinetic term of the action this means that $\dot A_0^2$ enters the Lagrangian with the negative sign,

$$S[\,A_\mu(x)\,]=\int d^4x\,\Big(-\frac12\eta^{\mu\nu}\partial_\mu A_\alpha\partial_\nu A_\beta\eta^{\alpha\beta}+\dots\Big)=\int d^4x\,\Big(\frac12\dot{\boldsymbol A}^2\underbrace{-\frac12\dot A_0^2}_{\text{ghost}}+\dots\Big), \tag{1.13}$$

so that the energy of the zeroth component of the vector field is not positive definite. This implies instabilities and leads to inconsistent theory both at the classical and quantum levels. The way to circumvent this difficulty is to use the property of *local gauge invariance* which excludes the "bad" mode from the spectrum of all physical modes. Let us demonstrate this property on the examples of mechanical relativistic particle and two field systems – electromagnetic and Yang-Mills fields.

1.3 Relativistic particle

The Lagrangian and the action of a relativistic particle moving in spacetime of $q^i \equiv x^\mu$ along the trajectory $x^\mu(t)$ read

$$L(x,\dot{x}) = -m\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} \equiv -m\sqrt{-\dot{x}^2}, \tag{1.14}$$

$$S[\,x(t)\,] = -m\int dt\sqrt{-\dot{x}^2}. \tag{1.15}$$

The action is invariant under one-dimensional diffeomorphism $t \mapsto t' = t'(t)$, $x(t) \mapsto x'(t') = x(t)$, which in the infinitesimal form can be written down as

$$t' = t + f(t), \quad \Delta^f x^\mu(t) \equiv x'^\mu(t) - x^\mu(t) = x'^\mu(t') - f(t)\dot{x}^\mu(t) - x^\mu(t) + O(f^2) = -f(t)\dot{x}^\mu(t), \tag{1.16}$$

where $f(t)$ is a small parameter of the transformation arbitrarily depending on time. This arbitrary time dependence means that this transformation is *local*. Such transformations we will call local gauge transformations. Note that in the definition of $\Delta^f x^\mu(t)$ we compare $x^\mu(t)$ and $x'^\mu(t)$ at one and the same value of the time parameter. As we will see now this local gauge invariance leads to peculiar canonical formalism of the theory.

One can directly check that the condition of the invertibility (1.3) of the matrix $\partial^2 L/\partial\dot{x}^\mu\partial\dot{x}^\nu$ is violated, so that the original velocities $\dot{x}^\mu$ cannot be expressed as functions of the momenta

$$p_\mu = \eta_{\mu\nu}\frac{m\dot{x}^\nu}{\sqrt{-\dot{x}^2}}, \tag{1.17}$$

and the momenta themselves are not independent and satisfy the identity $p^2 + m^2 = 0$, $p^2 \equiv \eta^{\mu\nu}p_\mu p_\nu$. We will call the left hand side of this identity the constraint function or simply the *constraint*,

$$T(p) = p^2 + m^2. \tag{1.18}$$

Moreover, the Hamiltonian turns out to be numerically identcally vanishing, $p_\mu\dot{x}^\mu - L = 0$, even though we cannot a priori express it in terms of coordinates and momenta (because the velocities are not expressible as functions of momenta). Therefore, the canonical action of the form (1.5) with $H = 0$ is unlikely to lead to the correct equations of motion. This is obvious because the momenta are subject to the constraint $T(p) = 0$ and cannot be varied as independent variables. This difficulty can be circumvented by considering the conditional variational principle by including into the action the constraint function with an arbitrary Lagrange multiplier N,

$$S[\,x,p,N\,] = \int dt(p_\mu\dot{x}^\mu - NT(p)), \tag{1.19}$$

and varying the total action with respect to the full set of x^μ, p_μ and N. This gives the set of equations

$$\dot{p}_\mu = 0, \quad \dot{x}^\mu - 2N\eta^{\mu\nu}p_\nu = 0, \quad T(p) = 0. \tag{1.20}$$

Substitution of the solution of the second equation $p_\mu = \dot{x}_\mu/2N$ into the constraint allows one to find N as $N = \pm\sqrt{-\dot{x}^2}$ and recover from the first of these equations the original variational equations for the Lagrangian action of the relativistic particle (1.15). On the other hand, the canonical action $S[\,x,p\,]$ with the Lagrangian value of the momentum leads to another form of the Lagrangian action as a functional of $x(t)$ and $N(t)$,

$$S_L[\,x(t),N(t)\,] \equiv S[\,x,\dot{x}/2N\,] = \int dt\,N\left(\frac{1}{4N^2}\dot{x}^\mu\dot{x}^\nu\eta_{\mu\nu} - m^2\right). \tag{1.21}$$

This action ia again invariant under the local one-dimensional diffeomorphism of time with $x^\mu \mapsto x'^\mu(t')$ given above and the Lagrange multiplier transforming as

$$x'^\mu(t') = x^\mu(t), \quad N'(t') = \left(\frac{dt'}{dt}\right)^{-1} N(t). \tag{1.22}$$

Note that with this definition one can interpret $Ndt = N'dt'$ as a differential of invariant (proper) time of the relativistic particle and $d/Ndt = d/N'dt'$ as the relevant invariant proper-time derivative. Correspondingly, the momentum $p_\mu = dx^\mu/2Ndt$ is just this proper-time derivative of the coordinate variable, which is invariant under these transformations $p_\mu = p'_\mu$. In the infinitesimal form the transformation of N and p_μ looks as

$$\Delta^f N(t) = N'(t) - N(t) = -\frac{d}{dt}(Nf), \quad \Delta^f p_\mu = 0. \tag{1.23}$$

Direct observation then shows that the Lagrangian gauge transformations of coordinates $\Delta^f x^\mu$ and Lagrange multiplier $\delta^f N$ can be imitated by the following set of transformations $\delta^{\mathcal{F}} x^\mu$, $\delta^{\mathcal{F}} p_\mu$ and $\delta^{\mathcal{F}} N$ in the canonical formalism with the action (1.19)

$$\delta^{\mathcal{F}} x^\mu = \{x^\mu, T\}\mathcal{F} = 2N\eta^{\mu\nu}p_\nu, \quad \delta^{\mathcal{F}} p_\mu = \{p_\mu, T\}\mathcal{F} = 0 \tag{1.24}$$

$$\delta^{\mathcal{F}} N = \dot{\mathcal{F}} \tag{1.25}$$

provided the Lagrangian parameter f and the canonical parameter $\mathcal{F}$ are related by the equation

$$\mathcal{F} = -Nf. \tag{1.26}$$

Then the Lagrangian transformations can be directly derived from the canonical ones on the equation of motion for the canonical momentum (or *on shell* of the momentum p_μ)

$$\Delta^f x^\mu = \delta^{\mathcal{F}} x^\mu \big|_{p=\dot{x}/2N}, \quad \Delta^f N = \delta^{\mathcal{F}} N. \tag{1.27}$$

Important difference between the transformations of the phase space variables (1.24) and those of the Lagrange multiplier (1.25) is that the first ones are the canonical transformations generated by the Poisson bracket with the constraint, whereas the second one is not canonical and involves the time derivative of the gauge parameter $\mathcal{F}$. It is easy to show by integration by parts in time that the infinitesimal transformations (1.24)-(1.25) leave the canonical action (1.19) invariant up to possible total derivative terms at the upper and lower limits of integration time range. So the presence of constraints in the canonical formalism is tightly related to local gauge invariance of the theory.

1.4 Electromagnetic and Yang-Mills fields

The action of the electromagnetic (EM) field, $q^i \mapsto A_\mu(\mathbf{x})$, $i = \mu, \mathbf{x}$, $\mu = 0, 1, 2, 3$, reads as

$$S[A_\mu(\mathbf{x})] = -\frac{1}{4}\int d^4x\, F_{\mu\nu}^2, \quad F_{\mu\nu}^2 = F_{\mu\nu}F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{1.28}$$

It is invariant under the gradient transformation with the scalar gauge parameter $f(x)$,

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu f(x), \quad \Delta^f A_\mu(x) = \partial_\mu f(x). \tag{1.29}$$

For the Lagrangian given by the integral over 3-dimensional space,

$$L = -\frac{1}{4}\int d^3x\, F_{\mu\nu}^2, \tag{1.30}$$

the canonical momenta are defined by 3-dimensional variational derivatives with respect to time derivatives of the vector potential, which also leads to the constraint — vanishing zeroth component of the momentum,

$$p^\mu(\mathbf{x}) = \frac{\delta L}{\delta \dot{A}_\mu(\mathbf{x})} = F^{\mu 0}, \quad p^0(\mathbf{x}) = 0. \tag{1.31}$$

As $p^0 = 0$ it is enough to disentangle from the Lagrangian the symplectic term for the nonvanishing components of the momentum $p^i \dot{A}_i$ in order to convert the action to the canonical form

$$S[\,A_\mu(x)\,] = \int dt\, d^3x \left(-\frac{1}{4}F_{ij}^2 + \frac{1}{2}F_{i0}^2\right)$$
$$= \int dt\, d^3x \left(p^i \dot{A}_i - \frac{1}{2}p_i^2 - \frac{1}{4}F_{ij}^2 - A_0(-\partial_i p^i)\right) = S[\boldsymbol{A}, \boldsymbol{p}, A_0]. \tag{1.32}$$

Here the role of the Hamiltonian is now played by the integral of the energy density of electromagnetic field $\frac{1}{2}\int d^3x\,(\boldsymbol{E}^2 + \boldsymbol{H}^2)$ in terms of electric $\boldsymbol{E} = E_i \equiv p_i$ and magnetic $\boldsymbol{H} = H_i \equiv \frac{1}{2}\epsilon_{ijk}F_{jk}$ field strengths, plus the term linear in A_0. The component A_0 enters linearly and does not involve a time derivative, so it plays, similarly to N in the relativistic particle case, the role of the Lagrange multiplier for the constraint on the momentum p_i. This constraint follows from the variation of the action with respect to $A_0(x)$,

$$\frac{\delta S[\boldsymbol{A}, \boldsymbol{p}, A_0]}{\delta A_0} = -T(p) = 0, \quad T(p) \equiv \partial_i p^i = 0, \tag{1.33}$$

and implies the nondynamical equation on the electric field strength in electrodynamics without electric charges $\mathrm{div}\,\boldsymbol{E} = 0$.

Like for the relativistic particle case, these constraints generate by the Poisson brackets the gauge transformations of phase space variables, corresponding to the gradient transformations in the Lagrangian formalism (1.29), whereas the transformation of the Lagrangian multiplier is given by the time derivative of the gauge parameter $\mathcal{F} = f$ (in this model gauge parameters in both formalisms simply coincide),

$$\begin{aligned}
\delta^{\mathcal{F}} A_i(\mathbf{x}) &= \left\{A_i(\mathbf{x}), \int d^3y\, T(\mathbf{y})\mathcal{F}(\mathbf{y})\right\} \\
&= \left\{A_i(\mathbf{x}), \int d^3y\, p^k(\mathbf{y})\partial_k \mathcal{F}_k(\mathbf{y})\right\} \\
&= \int d^3y\, \{A_i(\mathbf{x}), p^k(\mathbf{y})\}\, \partial_k \mathcal{F}(\mathbf{y}) = \partial_i \mathcal{F}(\mathbf{x}) = \Delta^f A_i, \\
\delta^{\mathcal{F}} p^i(\mathbf{x}) &= \left\{p^i(\mathbf{x}), \int d^3y\, T(\mathbf{y})\mathcal{F}(\mathbf{y})\right\} = 0, \\
\delta^{\mathcal{F}} A_0(\mathbf{x}) &= \dot{\mathcal{F}}(\mathbf{x}) = \Delta^f A_0.
\end{aligned} \tag{1.34}$$

Here we took into account that in the field-theoretical model the Poisson bracket of Eq.(1.7)) should be generalized to the expression

$$\{\mathcal{O}_1, \mathcal{O}_2\} = \frac{\partial \mathcal{O}_1}{\partial q^i}\frac{\partial \mathcal{O}_2}{\partial p_i} - \frac{\partial \mathcal{O}_1}{\partial p_i}\frac{\partial \mathcal{O}_2}{\partial q^i} \mapsto \int d^3x \left(\frac{\delta \mathcal{O}_1}{\delta A_i(\mathbf{x})}\frac{\delta \mathcal{O}_2}{\delta p^i(\mathbf{x})} - \frac{\delta \mathcal{O}_1}{\delta p^i(\mathbf{x})}\frac{\delta \mathcal{O}_2}{\delta A_i(\mathbf{x})}\right) \tag{1.35}$$

(in other words, if $i \mapsto (i, \mathbf{x})$ then $\sum_i \mapsto \sum_i \int d^3x$). Consequently

$$\{q^i, p_j\} = \delta^i_j \mapsto \{A_i(\mathbf{x}), p^k(\mathbf{y})\} = \delta^k_i \delta(\mathbf{x}, \mathbf{y}), \tag{1.36}$$

where the delta-function is defined by the following relation valid for any continuous test function $\varphi(\mathbf{x})$

$$\int d^3y\, \delta(\mathbf{x}, \mathbf{y})\, \varphi(\mathbf{y}) = \varphi(\mathbf{x}). \tag{1.37}$$

The case of electromagnetic field can be directly generalized to the Yang-Mills theory with the vector potential A^a_μ carrying extra color index a of the generating group algebra, say $SU(2)$ algebra with $a = 1, 2, 3$

$$A_\mu(x) \mapsto A^a_\mu(x), \qquad F_{\mu\nu} \mapsto F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + c^a_{bc} A^b_\mu A^c_\nu. \tag{1.38}$$

Here $c^a_{bc} = -c^a_{cb}$ are structure constants of this algebra satisfying the cyclic Jacobi identity

$$c^a_{bd} c^d_{ce} + \mathrm{cycle}(b, c, e) = 0, \tag{1.39}$$

which themselves can be considered as matrices of algebra generators $\hat{c}_d$ in the adjoint representation, satisfying the commutation relations,

$$c^a_{db} = (\hat{c}_d)^a_{\ b}\,, \qquad [\hat{c}_a, \hat{c}_b] = c^c_{ab}\hat{c}_c. \tag{1.40}$$

They determine symmetric Killing metric on the group algebra, $\gamma_{ab} = -\mathrm{tr}(\hat{c}_a\hat{c}_b)$, and fully antisymmetric tensor $c_{dab} = \gamma_{dc}c^c_{ab}$, which allows one to construct the gauge invariant Yang-Mills action

$$S_{\rm YM}[\,A^a_\mu(x)\,] = -\frac{1}{4}\int d^4x\,\gamma_{ab}F^a_{\mu\nu}F^{b\,\mu\nu}. \tag{1.41}$$

This action is invariant under the gauge transformations with the parameters f^a of the form expressible in terms of the covariant derivatives $\mathscr{D}_\mu$ with respect to the Yang-Mills fibre bundle connection

$$\Delta^f A^a_\mu = \partial_\mu f^a + A^b_\mu c^a_{bc} f^c \equiv \mathscr{D}_\mu\, f^a. \tag{1.42}$$

The canonical action has the form similar to the electromagnetic case,

$$S[A^a_\mu] = \int dt\int d^3x\left\{p^i_a\dot{A}^a_i - \frac{1}{2}(p^i_a)^2 - \frac{1}{4}\left(F^a_{ij}\right)^2 - A^a_0 T_a\right\}, \tag{1.43}$$

where for brevity we imply that the quadratic forms are determined with respect to the Killing metric γ_{ab} or its inverse γ^{ab}, $(p^i_a)^2 \equiv \gamma^{ab}p^i_a p^i_b$, $\left(F^a_{ij}\right)^2 = \gamma_{ab}F^a_{ij}F^b_{ij}$, and A^a_0 play the role of Lagrange multipliers of the constraints on the canonical momenta

$$T_a = -\mathscr{D}_i\, p^i_a \equiv -\left(\partial_i p^i_a - A^d_i c^b_{da} p^i_b\right). \tag{1.44}$$

Problem 1.1. Show that the canonical gauge transformation $\delta^{\mathcal{F}} A^a_i(\mathbf{x}) = \left\{A^a_i(\mathbf{x}), \int d^3y\, T_a(\mathbf{y})\mathcal{F}^a(\mathbf{y})\right\}$ with this generator gives the Lagrangian gauge transformation (1.42) for $\mathcal{F}^a = f^a$. Derive the relevant gauge transformation $\delta^{\mathcal{F}} p^i_a$.

There is essential difference of Yang-Mills gauge transformations from the gradient transformations in electrodynamics. Due to Abelian nature of EM field, gradient transformations are commuting, $[\Delta^{f_1}, \Delta^{f_2}] = 0$, so is the commutator of two canonical transformations determined by the Poisson bracket commutator of EM constraints, $\{T(\mathbf{x}), T(\mathbf{y})\} = 0$. The last relation is obvious because for electromagnetism the constraint $T(\mathbf{x})$ is A_i-independent. For non-Abelian Yang-Mills theory both commutators are nonvanishing.

Problem 1.2. Derive the commutators $[\Delta^{f_1}, \Delta^{f_2}]$ and $\{T_a(\mathbf{x}), T_b(\mathbf{y})\}$ in Yang-Mills theory and show their compatibility.

Lecture 2. Canonical condensed notations and reminder on Einstein gravity theory

- Canonical condensed DeWitt notations.
- Einstein gravity: a reminder

2.1 Canonical condensed DeWitt notations

Let us introduce condensed DeWitt notations. In transition from notations for a generic mechanical system to concrete relativistic particle, scalar field, EM and YM field models and gravity theory we have

$$q^i(t) = \underbrace{x^\mu(t)}_{i=\mu},\ \underbrace{\phi(t,\mathbf{x})}_{i=\mathbf{x}},\ \underbrace{A_\mu(t,\mathbf{x})}_{i=\mu,\mathbf{x}},\ \underbrace{A^a_\mu(t,\mathbf{x})}_{i=\mu,a,\mathbf{x}}\,\underbrace{g_{\mu\nu}(x)}_{i=\mu\nu,\mathbf{x}}, \tag{2.1}$$

where the index i absorbs now together with discrete labels also the continuous spatial coordinate $\mathbf{x}$. Specifically for YM we have the canonical coordinates and momenta

$$\begin{aligned} q^i &= A^a_i(\mathbf{x}), \quad p_i = p^i_a(\mathbf{x}), \quad i \longmapsto a, i, \mathbf{x} \\ T_\mu &= T_a(\mathbf{x}), \quad \mu \longmapsto a, \mathbf{x}, \end{aligned} \tag{2.2}$$

along with the notation for the constraints labelled by the condensed index μ accumulating again the discrete gauge transformations labels a (color indices of generating YM group) and coordinates $\mathbf{x}$.

Let us also extend these notations by the summation-integration rule – over contracted condensed indices we will assume both summation over discrete labels and integration over space

$$\begin{aligned} q^i p_i &= \int d^3x\, A_i^a(\mathbf{x}) p_a^i(\mathbf{x}), \\ \mathcal{F}^\mu T_\mu &= \int d^3x \mathcal{F}^a(\mathbf{x}) T_a(\mathbf{x}). \end{aligned} \tag{2.3}$$

Obviously this is the extension of the well-known Einstein rule of dropping the summation sign. We will call these notations the *canonical condensed* ones, when the time coordinate stays outside of the condensed label. This is of course the artifact of canonical formalism in which the time coordinate and the time derivative should be kept explicitly. Later we will also need the *covariant condensed* notations when the time will be also included into the condensed indices, the summation over them including the time integration.

The problems of the previous lecture are much easier to solve by using instead of the local constraints $T_a(\mathbf{x})$ their integrals with arbitrary test functions $\mathcal{F}_1^a(\mathbf{x})$ and $\mathcal{F}_2^a(\mathbf{x})$, which is much easier to operate with in terms of condensed notations. This looks as the following sequence of identical transformatioms,

$$\begin{aligned} \{T_a(x), T_b(y)\} &\to \int dx\, dy \mathcal{F}_1^a(\mathbf{x})\, \{T_a(\mathbf{x}), T_b(\mathbf{y})\}\, \mathcal{F}_2^b(\mathbf{y}) \\ &= \int dz dx dy \left(\mathcal{F}_1^a(x) \frac{\delta T_a(\mathbf{x})}{\delta A_i^c(\mathbf{z})} \frac{\delta T_b(\mathbf{y})}{\delta p_c^i(\mathbf{z})} \mathcal{F}_2^b(y) - (1 \leftrightarrow 2) \right) \\ &= \mathcal{F}_1^\mu \frac{\partial T_\mu}{\partial q^i} \frac{\partial T_\nu}{\partial p_i} \mathcal{F}_2^\nu - (1 \leftrightarrow 2) = \delta_q (\mathcal{F}_1^\mu T_\mu) \Big|_{\delta q^i = \frac{\partial T_\nu}{\partial p_i} \mathcal{F}_2^\nu} - (1 \leftrightarrow 2). \end{aligned} \tag{2.4}$$

$$\mathcal{F}^\mu T_\mu = \int dx\, \mathcal{F}^a \left(-\mathscr{D}_i\, p_a^i \right) = \int dx\, (\mathscr{D}_i \mathcal{F}^a)\, p_a^i, \tag{2.5}$$

$$\delta q^i = \frac{\partial\, (T_\nu \mathcal{F}_2^\nu)}{\partial p_i} = \frac{\delta}{\delta p_c^i(\mathbf{z})} \int dy\, (\mathscr{D}_k \mathcal{F}_2^b)\, p_b^k = \mathscr{D}_i\, \mathcal{F}_2^c(\mathbf{z}), \tag{2.6}$$

$$\delta_q\, (\mathcal{F}_1^\mu T_\mu) = \int dx\, \delta A_i^c\, c_{cb}^a\, \mathcal{F}_1^b\, p_a^i \Big|_{\delta A = \mathscr{D} \mathcal{F}_2} = \int dx\, (\mathscr{D}_i \mathcal{F}_2^c)\, c_{cb}^a\, \mathcal{F}_1^b\, p_a^i \tag{2.7}$$

Hence

$$\begin{aligned} \int dx\, dy \mathcal{F}_1^a(\mathbf{x})\, \{T_a(\mathbf{x}), T_b(\mathbf{y})\}\, \mathcal{F}_2^b(\mathbf{y}) &= \int dx\, (\mathscr{D}_i \mathcal{F}_2^c)\, c_{cb}^a\, \mathcal{F}_1^b\, p_a^i - (1 \leftrightarrow 2) \\ &= \int dx\, \mathscr{D}_i (\mathcal{F}_2^c\, c_{cb}^a\, \mathcal{F}_1^b)\, p_a^i = - \int dx \mathcal{F}_1^a c_{ab}^d T_d \mathcal{F}_2^b, \end{aligned} \tag{2.8}$$

or

$$\{T_a(\mathbf{x}), T_b(\mathbf{y})\} = -c_{ab}^d T_d(\mathbf{x}) \delta(\mathbf{x}, \mathbf{y}), \tag{2.9}$$

which in condensed notations can be rewritten as

$$\begin{aligned} &\{T_\mu, T_\nu\} = \mathcal{U}_{\mu\nu}^\lambda T_\lambda, \quad \mathcal{U}_{\mu\nu}^\lambda = -c_{ab}^d \delta(\mathbf{z}, \mathbf{x})\, \delta(\mathbf{z}, \mathbf{y}), \\ &\lambda \mapsto d\, \mathbf{z}, \quad \mu \mapsto a\, \mathbf{x}, \quad \nu \mapsto b\, \mathbf{y}. \end{aligned} \tag{2.10}$$

Thus, non-Abelian nature of the YM gauge transformation for spatial components of the vector potential,

$$(\Delta^{f_2} \Delta^{f_1} - \Delta^{f_1} \Delta^{f_2}) A_i^a = \Delta^{f_3} A_i^a, \quad f_3^a = c_{bc}^a f_2^b f_1^c \tag{2.11}$$

fully matches with the non-Abelian canonical transformations in the canonical formalism, because

$$\begin{aligned} &(\delta^{\mathcal{F}_2} \delta^{\mathcal{F}_1} - \delta^{\mathcal{F}_1} \delta^{\mathcal{F}_2}) A_i^a(\mathbf{x}) = \{\{A_i^a(\mathbf{x}), T_\mu\}, T_\nu\} \mathcal{F}_2^\mu\, \mathcal{F}_1^\nu - (1 \leftrightarrow 2) \\ &\qquad\qquad = \{A_i^a(\mathbf{x}), \{T_\mu, T_\nu\}\} \mathcal{F}_2^\mu\, \mathcal{F}_1^\nu = \delta^{\mathcal{F}_3} A_i^a(\mathbf{x}), \\ &\mathcal{F}_3^\lambda = -\mathcal{U}_{\mu\nu}^\lambda \mathcal{F}_2^\mu \mathcal{F}_1^\nu, \end{aligned} \tag{2.12}$$

where we used the Poisson bracket Jacobi identity $\{\{F_1, F_2\}, F_3\} + \mathrm{cycle}(1,2,3) = 0$ and the constraint algebra (2.10).

The gauge transformation of the remaining zeroth component of the vector potential $\delta^{\mathcal{F}} A^a_0(\mathbf{x}) = \Delta^f A^a_0(\mathbf{x}) \equiv \mathscr{D}_0\,\mathcal{F}^a(\mathbf{x}) = \dot{\mathcal{F}}^a(\mathbf{x}) + c^a_{bd} A^b_0(\mathbf{x})\,\mathcal{F}^d(\mathbf{x})$, can be written down in the form

$$\delta^{\mathcal{F}} \lambda^\mu = \dot{\mathcal{F}}^\mu - \mathcal{U}^\mu_{\alpha\beta} \lambda^\alpha \mathcal{F}^\beta \tag{2.13}$$

if we introduce the special notation for $A^a_0(\mathbf{x})$ as the Lagrange multiplier $\lambda^\mu \equiv A^a_0(\mathbf{x})$. The meaning and generality of this representation we will see later after we consider the case of gravity theory.

2.2 Einstein gravity: a reminder

Einstein gravity theory with matter fields has the action

$$S[\,g_{\mu\nu},\phi\,] = \frac{1}{16\pi G}\int d^4x\, g^{1/2}\,(R - 2\Lambda) + \text{surface term} + S_{\rm m}[\,\phi, g_{\mu\nu}\,], \tag{2.14}$$

where R is the curvature scalar of the metric $g_{\mu\nu}(x)$, $g = -\det g_{\mu\nu}$, $\phi(x)$ is the set of matter fields, G and Λ are the gravitational and cosmological constants and for a time beiong we do not specify the surface term of the gravitational action. The action is invariant under local diffeomorphisms

$$x^\mu \to x^{\mu'} = x^{\mu'}(x), \tag{2.15}$$

under which the metric tensor and matter field tensor $\phi(x)$ transform as

$$g_{\mu'\nu'}(x') = \frac{\partial x^\alpha}{\partial x^{\mu'}}\frac{\partial x^\beta}{\partial x^{\nu'}}\, g_{\alpha\beta}(x), \tag{2.16}$$

$$\phi(x) \mapsto \phi'(x') = \hat{D}\left(\frac{\partial x'}{\partial x}\right)\phi(x). \tag{2.17}$$

Here $\hat{D}\,(\partial x'/\partial x) = D^A_B\,(\partial x'/\partial x)$ is the matrix of the representation of the general linear group GL(4) to which belongs the field $\phi = \phi^A$. A and B here are generic spin-tensor indices, and the matrices acting in the vector space of these indices will be denoted by hat.

This matrix $\hat{D}\,(\partial x'/\partial x)$ is parameterized by the elements of the Jacobi matrix of transition to new coordinates $\partial x'/\partial x$. For example, in case of contravariant vector field $\phi(x) = A^\mu(x)$, $A^{\mu'}(x') = (\partial x^{\mu'}/\partial x^\nu)A^\nu(x)$, $\hat{D}\,(\partial x'/\partial x) = (\partial x^{\mu'}/\partial x^\nu)$. For infinitesimal diffeomorphism $x^{\mu'} = x^\mu + f^\mu(x)$ these transformations express in terms of the Lie derivative along the vector field f^μ,

$$\begin{aligned}\Delta^f g_{\mu\nu}(x) &= g'_{\mu\nu}(x) - g_{\mu\nu}(x) = -\pounds_f\, g_{\mu\nu}\\ &= -f^\alpha\partial_\alpha g_{\mu\nu} - \partial_\mu f^\alpha g_{\alpha\nu} - \partial_\nu f^\alpha g_{\alpha\mu} = -\nabla_\mu f_\nu - \nabla_\nu f_\mu,\\ \Delta^f \phi(x) &= \phi'(x) - \phi(x) = -\pounds_f\,\phi\\ &= -f^\mu\partial_\mu\phi + \partial_\mu f^\nu \hat{G}^\mu_\nu \phi = -f^\mu\nabla_\mu\phi + \nabla_\mu f^\nu \hat{G}^\mu_\nu\phi.\end{aligned} \tag{2.18}$$

Here $\hat{G}^\mu_\nu$ are the generators of the GL(4)-representation parameterizing the group matrix in the vicinity of the identical transformation

$$\hat{D}\left(1 + \frac{\partial f}{\partial x}\right) = \hat{1} + \partial_\mu f^\nu \hat{G}^\mu_\nu, \tag{2.19}$$

and ∇_μ denotes the covariant derivative with respect to the Christoffel connection

$$\nabla_\mu\phi = \partial_\mu\phi + \Gamma^\alpha_{\mu\beta}\hat{G}^\beta{}_\alpha\phi, \quad \nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\alpha}A^\alpha, \tag{2.20}$$

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\lambda}\left(\partial_\alpha g_{\lambda\beta} + \partial_\beta g_{\lambda\alpha} - \partial_\lambda g_{\alpha\beta}\right), \quad \nabla_\mu g_{\alpha\beta} = 0. \tag{2.21}$$

The Riemann and Ricci tensors are defined as

$$(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)\phi^\alpha = R^\alpha{}_{\beta\mu\nu}\phi^\beta, \quad R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}. \tag{2.22}$$

In what follows we will need to perform metric variations. They can be done via the solution of the following problem

Problem 2.1. Derive the following variations under the infinitesimal variation of the metric tensor $\delta g_{\mu\nu} \equiv h_{\mu\nu}$, assuming that by definition $h^\mu_\nu \equiv g^{\mu\alpha} h_{\alpha\nu}$, $h^{\mu\nu} \equiv g^{\mu\alpha} g^{\nu\beta} h_{\alpha\beta}$,

$$\delta(g^{\mu\nu}) = -h^{\mu\nu}, \quad \delta g^{1/2} = \frac{1}{2}\, g^{1/2} g^{\mu\nu} h_{\mu\nu} \equiv \frac{1}{2}\, g^{1/2} h,$$
$$\delta \Gamma^\alpha_{\mu\nu} = \frac{1}{2} \left(\nabla_\mu h^\alpha_\nu + \nabla_\nu h^\alpha_\mu - \nabla^\alpha h_{\mu\nu}\right),$$
$$\delta R^\alpha{}_{\beta\mu\nu} = \nabla_\mu (\delta \Gamma^\alpha_{\nu\beta}) - \nabla_\nu (\delta \Gamma^\alpha_{\mu\beta}).$$

The metric variation of the gravitational part of the full action – Einstein-Hilbert action $S_{\rm EH}[\,g_{\mu\nu}\,]$ equals up to a contribution of the surface term

$$\begin{aligned} \delta_g S_{\rm EH}[\,g_{\mu\nu}\,] &= \frac{1}{16\pi G} \int_{^4\mathcal{M}} d^4x\, g^{1/2} \left(\frac{1}{2} g^{\mu\nu} h_{\mu\nu}\,(R - 2\Lambda) + (\delta g^{\mu\nu})\, R_{\mu\nu} + g^{\mu\nu} \delta R^\alpha{}_{\mu\alpha\nu}\right) \\ &= -\frac{1}{16\pi G} \int d^4x\, g^{1/2}\, (G^{\mu\nu} + \Lambda g^{\mu\nu})\, \delta g_{\mu\nu}, \end{aligned} \tag{2.23}$$

where $G^{\mu\nu}$ is the Einstein tensor which satisfies a well-known contracted Bianchi identity

$$G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R, \qquad \nabla_\mu G^{\mu\nu} \equiv 0. \tag{2.24}$$

This identity is obviously compatible with the invariance of the action under diffeomorpisms, because by integrating by parts and using the compact support of f^μ one has

$$\Delta^f S_{\rm EH} = \frac{1}{16\pi G} \int d^4x\, g^{1/2}\, (G^{\mu\nu} + \Lambda g^{\mu\nu})\, (\nabla_\mu f_\nu + \nabla_\nu f_\mu) = -\frac{1}{16\pi G} \int d^4x\, g^{1/2} 2\, (\nabla_\mu G^{\mu\nu})\, f_\nu = 0. \tag{2.25}$$

For minimally interacting with gravity matter fields their action in curved spacetime follows from the flat space one by the replacement of the metric by the curved metric, the replacement of integration measure by the Riemanninan measure and trading the partial spacetime derivatives for covariant derivatives

$$\eta_{\mu\nu} \mapsto g_{\mu\nu}, \quad \partial_\mu \mapsto \nabla_\mu, \quad \int d^4x \mapsto \int d^4x\, g^{1/2}. \tag{2.26}$$

In view of diffeomorphism invariance of matter action its metric stress tensor

$$T^{\mu\nu} = \frac{2}{g^{1/2}} \frac{\delta S_{\rm m}}{\delta g_{\mu\nu}} \tag{2.27}$$

is covariantly conserved on the solution of equations of motion for matter fields

$$0 = \Delta^f S_{\rm m}\,\Big|_{\frac{\delta S_{\rm m}}{\delta \Phi} = 0} = \int d^4x\, g^{1/2} T^{\mu\nu} \nabla_\mu f_\nu = -\int d^4x\, g^{1/2} \nabla_\mu T^{\mu\nu} f_\nu, \quad \nabla_\mu T^{\mu\nu} = 0. \tag{2.28}$$

Problem 2.2. Write down the action of relativistic particle in curved spacetime, derive the equations of motion of the particle, its metric stress tensor and prove that this tensor is conserved on equations of motion,

Lecture 3. Geometry of (3+1) spacetime foliation

- (3+1) spacetime foliation
- Time evolution of the spacetime hypersurface $\sigma(t)$
- Projections of Riemann tensor.

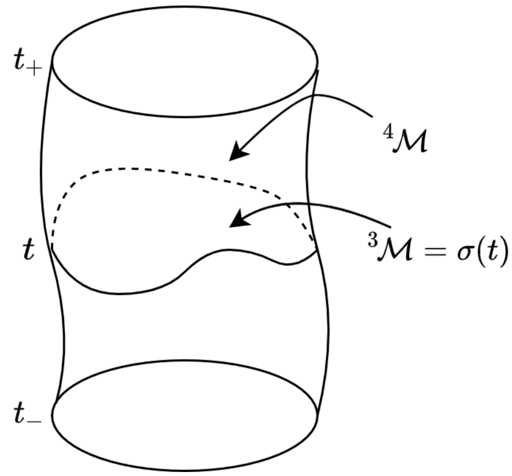

Figure 1: Spacetime foliation by hypersurfaces of constant time t.

3.1 (3+1) spacetime foliation

Canonical formalism of the gravitational field requires disentangling time from the full set of spacetime coordinates $x^\mu = (x^0, x^i)$. This can be done in a way preserving the initial 4-dimensional general coordinate invariance and the 3-dimensional one. The idea is to foliate the 4-dimensional spacetime $^4\mathcal{M} = [t_-, t_+] \times {}^3\mathcal{M}$ by spacelike hypersurfaces $\sigma(t) = {}^3\mathcal{M}$ of constant time t. This foliation can be described as embedding into the 4-dimensional spacetime $^4\mathcal{M}$ of coordinates $x^\alpha = (x^0, x^i)$ of the one-parameter family of surfaces parameterized by intrinsic coordinates $\mathbf{x} = x^a$, $a = 1, 2, 3$, and labelled by time t,

$$\sigma(t): \quad x^\alpha = e^\alpha(x^a, t). \tag{3.1}$$

Here we will use Greek letters to label 4-dimensional spacetime objects and letters from the first part of Latin alphabet for 3-dimensional objects on $\sigma(t)$. In this way we retain the covariance under both 4-dimensional diffeomorphisms and 3-dimensional diffeomorphisms explicitly depending on time t,

$$\begin{aligned} {}^4\mathcal{M}: &\quad x^\mu \to x^{\mu'} = x^{\mu'}(x^\nu) \\ {}^3\mathcal{M}: &\quad x^a \to x^{a'} = x^{a'}(x^b, t). \end{aligned} \tag{3.2}$$

The basis of three vectors tangential to $\sigma(t)$ can be written down as partial derivatives of embedding functions $e^\alpha(\mathbf{x}, t)$ with respect to x^a. Together with the vector n_α normal to the surface they form the *normal* basis

$$e^\alpha_a = \frac{\partial e^\alpha(\mathbf{x}, t)}{\partial x^a}, \qquad n_\alpha e^\alpha_a = 0, \qquad n_\alpha = g_{\alpha\beta} n^\beta, \tag{3.3}$$

while the metric interval along the surface with $dx^\alpha = e^\alpha_a dx^a$

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = \underbrace{e^\alpha_a g_{\alpha\beta} e^\beta_b}_{\gamma_{ab}} dx^a dx^b \tag{3.4}$$

suggests the notion of induced metric

$$\gamma_{ab} = e^\alpha_a g_{\alpha\beta} e^\beta_b. \tag{3.5}$$

The vector n_α normal to spacelike hypersurfaces is normalized to -1, but for generality we will use the notation $\epsilon = \pm 1$ for its norm to indicate whether it is timelike or spacelike

$$g_{\alpha\beta} n^\alpha n^\beta = \epsilon = \pm 1. \tag{3.6}$$

Every spacetime vector can be decomposed in the normal basis into its normal and tangential components labelled as it is shown here

$$\phi^\alpha(x)\Big|_{x=e(\mathbf{x},t)} = \phi_\perp n^\alpha + \phi^a e^\alpha_a. \tag{3.7}$$

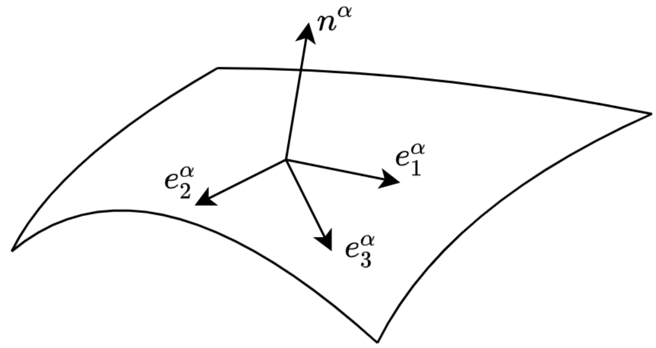

Figure 2: Normal basis $\{n^\alpha, e_a^\alpha\}$

Conversely its components read as projections on the vectors of the normal basis

$$\phi_\perp = \epsilon n_\alpha \phi^\alpha, \qquad \phi^a = e_\alpha^a \phi^\alpha, \tag{3.8}$$

where we use the contravariant induced metric γ^{ab}, $\gamma^{ab}\gamma_{bc} = \delta_b^a$, to raise 3-dimensional indices,

$$e_\alpha^a \equiv \gamma^{ab} e_b^\beta g_{\beta\alpha}. \tag{3.9}$$

From this definition it immediately follows the following contraction of indices

$$e_a^\alpha e_\alpha^b = \delta_a^b. \tag{3.10}$$

The decomposition of the spacetime metric in the normal basis leads in view of $g_{\perp\perp} = \epsilon$ and $g_{\perp a} = 0$ to the following relation

$$g_{\alpha\beta} = \epsilon n_\alpha n_\beta + \gamma_{ab} e_\alpha^a e_\beta^b, \tag{3.11}$$

whence it follows the definition of the tensor γ_β^α of projection onto the hypersurface

$$e_a^\alpha e_\beta^a = \delta_\beta^\alpha - \epsilon n^\alpha n_\beta \equiv \gamma_\beta^\alpha, \tag{3.12}$$

which satisfies the orthogonality relations

$$\gamma_\beta^\alpha n_\alpha = n^\beta \gamma_\beta^\alpha = 0. \tag{3.13}$$

The triad of vectors e_a^α obviously transforms under the both diffeomorphisms (3.2) as

$$e_a^\alpha \mapsto e_{a'}^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^\alpha} e_a^\alpha \frac{\partial x^a}{\partial x^{a'}}. \tag{3.14}$$

To introduce the spatial covariant derivative on the hypersurface, induced from the 4-dimensional space, which would have a correct transformation law under 3-dimensional diffeomorphisms, consider a 3-dimensional vector ϕ^b and raise it to the level of the 4-dimensional vector field defined on the hypersurface

$$\phi^b \mapsto \phi^\beta \equiv e_b^\beta \phi^b. \tag{3.15}$$

Then calculate its 4-dimensional covariant derivative along the hypersurface $\nabla_a \equiv e_a^\alpha \nabla_\alpha$ and project the result back to the hypersurface. One obtains

$$\mathscr{D}_a \phi^b = e_a^\alpha \nabla_\alpha (e_c^\beta \phi^c) e_\beta^b = \underbrace{\partial_a}_{e_a^\alpha \partial_\alpha} \phi^b + \underbrace{\left(\nabla_a e_c^\beta\right) e_\beta^b}_{\gamma_{ac}^b} \phi^c, \tag{3.16}$$

where the last term should be interpreted in terms of the 3-dimensional connection

$$\gamma_{ac}^b = (\nabla_a e_c^\alpha) e_\alpha^b = {}^{(3)}\Gamma_{ac}^b, \quad \nabla_a \equiv e_a^\alpha \nabla_\alpha, \tag{3.17}$$

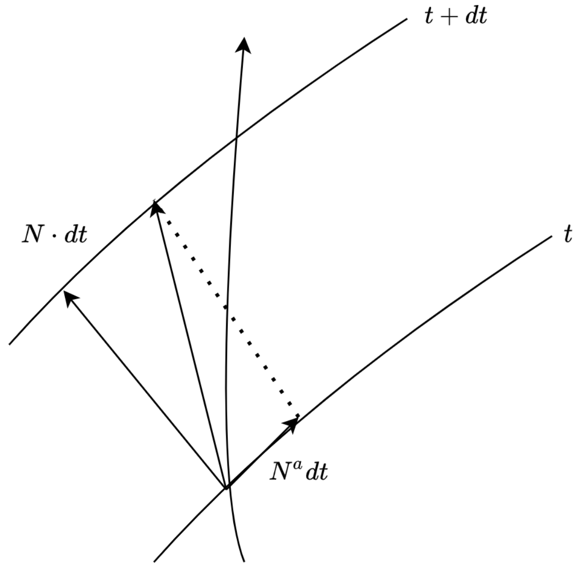

Figure 3: Lapse and shift functions N and N^a

which turns out to be the Christoffel symbol of γ_{ab}.

Problem 3.1. Prove that $\gamma^c_{ab} = (\nabla_a e^\alpha_b)\, e^c_\alpha$ is the the Christoffel symbol of γ_{ab}, because $\mathscr{D}_a \gamma_{bc} = 0$.

Extrinsic curvature is determined by the covariant derivative of the normal vector taken along the hypersurface. It is a symmetric tensor given by the following several equivalent expressions

$$K_{ab} \equiv -\nabla_\alpha n_\beta e^\alpha_a e^\beta_b = -\left(\nabla_a n_\beta\right) e^\beta_b = +n_\beta \left(\nabla_a e^\beta_b\right). \tag{3.18}$$

Eqs.(3.17) and (3.18) imply that they represent respectively the normal and tangential projections of $\nabla_a e^\alpha_b$ with respect to index α. Thus they give rise to Gauss-Weingarten formula

$$\nabla_a e^\alpha_b = \epsilon n^\alpha K_{ab} + \gamma^c_{ab} e^\alpha_c. \tag{3.19}$$

3.2 Time evolution of the spacetime hypersurface $\sigma(t)$

Time "evolution" of the hypersurface in $\sigma(t)$ can be described by its "velocity"

$$N^\alpha = \frac{\partial e^\alpha(\mathbf{x}, t)}{\partial t} \tag{3.20}$$

with which it evolves in spacetime. Its decomposition in the normal basis

$$N^\alpha = n^\alpha N + e^\alpha_a N^a \tag{3.21}$$

determines the *lapse* function N and *shift* functions N^a which in Fig.3 illustrate the movement of the point of fixed spatial coordinates $\mathbf{x}$ at the transition from the hypersurface $\sigma(t)$ to $\sigma(t+dt)$.

The tensor fields at the point of spacetime belonging to the hypersurface is obviously a functional of the embedding functions $\phi[\,e(\mathbf{x}\,]$, and one can define the covariant derivative in the direction of the vector N^α in terms of the variation of the embedding $\delta e^\alpha = \partial_t e^\alpha(\mathbf{x}, t) dt$

$$\boldsymbol{D}_N \phi \equiv \delta_e \phi / dt + N^\alpha\, \mathit{\Gamma}^\beta_{\alpha\lambda} \hat{G}^\lambda_{\ \beta} \phi, \tag{3.22}$$

where

$$\delta_e \phi[\,e\,]/dt = \int d^3 y\, \frac{\delta \phi[\,e(\mathbf{x})\,]}{\delta e^\alpha(\mathbf{y})}\, N^\alpha(\mathbf{y}, t) = \partial_t \phi[\,e(\mathbf{x}, t)\,] \tag{3.23}$$

is a partial derivative written in terms of the variation of $\phi[e(\mathbf{x}]$ with respect to the embedding variation $\delta e^\alpha = N^\alpha\, dt$. Apply this covariant derivative to e^α_a and make a sequence of transformations

$$\begin{aligned} \boldsymbol{D}_N e^\alpha_a &= \partial_t \left(e^\alpha_a\right) + \mathit{\Gamma}^\alpha_{\beta\lambda} e^\lambda_a\, N^\beta = \partial_a \left(\partial_t e^\alpha_a\right) + \mathit{\Gamma}^\alpha_{\beta\lambda} e^\lambda_a\, N^\beta \\ &= \partial_a (n^\alpha N + e^\alpha_b N^b) + (n^\beta N + e^\beta_b N^b) \mathit{\Gamma}^\alpha_{\beta\lambda} e^\lambda_a \\ &= \nabla_a n^\alpha N + n^\alpha \partial_a N + (\nabla_a e^\alpha_b) N^b + e^\alpha_b \partial_a N^b \\ &= -K_{ab} e^{\alpha b} N + e^\alpha_b \mathscr{D}_a\, N^b + n^\alpha (\mathscr{D}_a\, N + \epsilon K_{ab} N^b), \end{aligned} \tag{3.24}$$

where we used the fact that from the definition of the extrinsic curvature (3.18) $\nabla_a n^\alpha = -K_{ab} e^{\alpha b}$ (remember that $\nabla_\beta n^\alpha n_\alpha = 0$ in view of $n^\alpha n_\alpha = \epsilon$) and also collected partial derivative terms with Christoffel symbol terms to form the covariant derivative $\nabla_a e^\alpha_b$ for which we used the Gauss-Wengarten equation (3.19). The final result for the time evolution of the tangential basis e^α_a reads

$$\boldsymbol{D}_N e^\alpha_a = e^\alpha_b \left(\mathscr{D}_a\, N^b - K^b_a N\right) + n^\alpha \left(\mathscr{D}_a\, N + \epsilon K_{ab} N^b\right), \tag{3.25}$$

which allows one to express the time derivative of induced metric via extrinsic curvature

$$\frac{\partial \gamma_{ab}}{\partial t} = \partial_t \left(e^\alpha_a g_{\alpha\beta} e^\beta_b\right) = \boldsymbol{D}_N \left(e^\alpha_a e_{\alpha b}\right) = 2\boldsymbol{D}_N e^\alpha_{(a} e_{\alpha b)} = \mathscr{D}_a\, N_b + \mathscr{D}_b\, N_a - 2K_{ab} N. \tag{3.26}$$

As a result there is another hypostasis of the extrinsic curvature tensor in terms of the time derivative of the induced metric

$$K_{ab} = \frac{1}{2N} \left(\mathscr{D}_a\, N_b + \mathscr{D}_b\, N_a - \dot{\gamma}_{ab}\right). \tag{3.27}$$

Problem 3.2. Derive the equation for the evolution of the normal vector

$$\boldsymbol{D}_N n_\alpha = -\epsilon e^a_\alpha \partial_a N - e^a_\alpha K_{ab} N^b. \tag{3.28}$$

3.3 Projections of the Riemann tensor

It is useful to promote the 3-dimensional K_{ab} to the level of the 4-dimensiona ${}^4\mathcal{M}$-tensor by the following equation

$$K_{\mu\nu} \equiv e^a_\mu e^b_\nu K_{ab} = -\underbrace{e^a_\mu e^\alpha_a}_{\gamma^\alpha_\mu} \underbrace{e^b_\nu e^\beta_b}_{\gamma^\beta_\nu} \nabla_\alpha n_\beta = -\gamma^\alpha_\mu \gamma^\beta_\nu \nabla_\alpha n_\beta = -\gamma^\alpha_\mu \nabla_\alpha n_\nu. \tag{3.29}$$

This tensor is obviously orthogonal to the normal vector with respect to both of its symmetric indices

$$K_{\mu\nu} n^\nu = 0, \tag{3.30}$$

Let us use this tensor for the derivation of various projections of the 4-dimensional curvature onto the normal basis. For this purpose consider a generic 3-dimensional vector field ϕ^a and again lift it to the level of the 4-dimensional vector field having a zero normal component,

$$\phi^a \to \phi^\mu \equiv e^\mu_a \phi^a, \quad \phi^\mu n_\mu = 0. \tag{3.31}$$

Then by definition its covariant spatial derivative can be lifted to the level of 4-dimensional tensor by covariantly differentiating it and projecting the result with respect to both indices onto the hyoersurface

$$\mathscr{D}_\mu\, \phi^\nu = \gamma^\alpha_\mu \gamma^\nu_\beta \nabla_\alpha \phi^\beta. \tag{3.32}$$

Similar construction holds for the covariant derivative of higher rank tensors having vanishing normal components.

Consider the commutator of two covariant derivatives acting on the normal vector and project the result onto the surface. This can be transformed as

$$\begin{aligned} 2\gamma^\alpha_{[\mu}\gamma^\beta_{\nu]}\nabla_\alpha\nabla_\beta n^\lambda\gamma^\sigma_\lambda &= 2\gamma^\alpha_{[\mu}\nabla_\alpha\left(\gamma^\beta_{\nu]}\nabla_\beta n^\lambda\gamma^\sigma_\lambda\right) - 2\left(\gamma^\alpha_{[\mu}\nabla_\alpha\gamma^\beta_{\nu]}\right)\nabla_\beta n^\lambda\gamma^\sigma_\lambda + 2\left(\gamma^\alpha_{[\mu}\nabla_\alpha\gamma^\sigma_\lambda\right)K^\lambda_{\nu]} \\ &= -2\mathscr{D}_{[\mu}K^\sigma_{\nu]} + 2\epsilon n_{[\nu}K^\beta_{\mu]}K^\sigma_\beta, \end{aligned} \tag{3.33}$$

where square brackets denote antisymmetrization of *two* indices (with the 1/2 coefficient, $A_{[\mu\nu]} \equiv \frac{1}{2}(A_{\mu\nu}+A_{\nu\mu})$) and we used simple relations for the derivatives of the projection operators

$$\gamma^\alpha_\mu\nabla_\alpha\gamma^\sigma_\lambda = \epsilon(K^\sigma_\mu n_\lambda + n^\sigma K_{\mu\lambda}), \quad \gamma^\alpha_\mu\gamma^\beta_\nu\nabla_\alpha\gamma^\sigma_\beta = \epsilon n^\sigma K_{\mu\nu}. \tag{3.34}$$

On the other hand the left hand side of (3.33), being the commutator of two covariant derivatives, equals $\gamma^\alpha_\mu\gamma^\beta_\nu R^\lambda{}_{\delta\alpha\beta}\, n^\delta\gamma^\sigma_\lambda$, so that

$$\gamma^\sigma_\lambda R^\lambda{}_{\delta\alpha\beta}\, n^\delta\gamma^\alpha_\mu\gamma^\beta_\nu = -\mathscr{D}_\mu K^\sigma_\nu + \mathscr{D}_\nu K^\sigma_\mu + 2\epsilon n_{[\nu}K^\beta_{\mu]}K^\sigma_\beta. \tag{3.35}$$

Therefore, the following projection of the 4-dimensional Riemann tensor reads

$${}^{(4)}R_{\perp abc} = 2\epsilon\mathscr{D}_{[b}K_{c]a}. \tag{3.36}$$

Analogous projection of the commutator of two covariant derivatives acting on a generic vector fiels tangential to the surface yields the *Gauss-Codazzi* equation

$${}^{(4)}R_{abcd} = {}^{(3)}R_{abcd} - 2\epsilon K_{a[c}K_{d]b}. \tag{3.37}$$

Problem 3.3. Prove Eq.(3.34) and the Gauss-Codazzi equation.

For the calculation of ${}^{(4)}R = g^{\mu\nu}g^{\alpha\beta}R_{\mu\alpha\nu\beta} = 2\gamma^{ab}R_{a\perp b\perp} + {}^{(4)}R^{ab}{}_{ab}$ one would need a projection $R_{a\perp b\perp}$ which is rather complicated because it contains second order time derivatives of γ_{ab}. This difficulty can be circumvented by using the following transformations. From Gauss-Codazzi equation it follows that

$${}^{(4)}R^{ab}{}_{ab} = {}^{(3)}R - \epsilon(K^2 - K^2_{ab}), \tag{3.38}$$

where $K = \gamma^{ab}K_{ab}$ is the trace of the extrinsic curvature, and we used the abbreviation $K^2_{ab} = K^{ab}K_{ab}$. On the other hand

$$\begin{aligned} {}^{(4)}R_{abcd}\gamma^{ac}\gamma^{bd} &= {}^{(4)}R_{\mu\nu\alpha\beta}\gamma^{\mu\alpha}\gamma^{\nu\beta} \\ &= {}^{(4)}R_{\mu\nu\alpha\beta}(g^{\mu\alpha} - \epsilon n^\mu n^\nu)(g^{\nu\beta} - \epsilon n^\nu n^\beta) \\ &= {}^{(4)}R - 2\epsilon R_{\mu\nu}n^\mu n^\nu = -2\epsilon G_{\mu\nu}n^\mu n^\nu = -2\epsilon G_{\perp\perp}, \end{aligned} \tag{3.39}$$

so that the normal-normal projection of the Einstein tensor equals the expression

$$G_{\perp\perp} = \frac{1}{2}(K^2 - K^2_{ab} - \epsilon^3 R), \tag{3.40}$$

which does not at all involve second order time derivatives of γ_{ab} (see Eq.(3.18). Consequently, for the 4-dimensional scalar curvature one can write the following sequence of relations using the fact that Ricci tensor is expressible with the aid of the commutator of two covariant derivatives,

$$\begin{aligned} {}^{(4)}R &= 2\epsilon(R_{\mu\nu} - G_{\mu\nu})n^\mu n^\nu = 2\epsilon R_{\mu\nu}n^\mu n^\nu - 2\epsilon G_{\perp\perp} \\ &= 2\epsilon n^\nu(\nabla_\alpha\nabla_\nu - \nabla_\nu\nabla_\alpha)n^\alpha + {}^{(3)}R + \epsilon K^2_{ab} - \epsilon K^2 \\ &= 2\epsilon\nabla_\alpha(n^\nu\nabla_\nu n^\alpha - n^\alpha\nabla_\nu n^\nu) - 2\epsilon\underbrace{(\nabla_\alpha n^\nu)(\nabla_\nu n^\alpha)}_{K^2_{ab}} + 2\epsilon\underbrace{(\nabla_\alpha n^\alpha)^2}_{K^2} + {}^{(3)}R + \epsilon(K^2_{ab} - K^2), \end{aligned} \tag{3.41}$$

whence the 4-dimensional scalar curvature takes the form

$${}^{(4)}R = {}^{(3)}R - \epsilon(K^2_{ab} - K^2) + 2\epsilon\nabla_\mu(n^\nu\nabla_\nu n^\mu - n^\mu\nabla_\nu n^\nu). \tag{3.42}$$

Thus, up to the contribution of the total derivative term the scalar curvature is expressible as a combination of the 3-dimensional scalar curvature and the form quadratic in the extrinsic curvature.

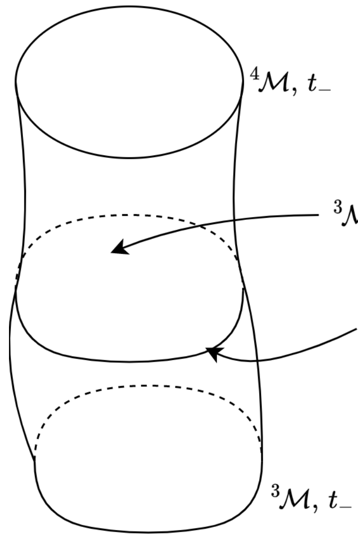

Figure 4: Spacetime domain in the canonical formalism of gravity theory.

Lecture 4. Canonical formalism of Einstein general relativity

- Arnowitt-Deser-Misner (ADM) varables and ADM action of GR.
- Canonical action and Hamiltonian and momenta constraints.
- Canonical gauge transformations.

4.1 Arnowitt-Deser-Misner (ADM) varables and ADM action of GR

Let us integrate the scalar curvature over the spacetime domain ${}^4\mathcal{M}$ depicted on Fig.4, whose total boundary,

$$\partial\,{}^4\mathcal{M} = {}^3\mathcal{M}(t_+) \cup {}^3\mathcal{M}(t_-) \cup \Big([t_-,t_+] \times \partial\,{}^3\mathcal{M}\Big), \tag{4.1}$$

consists of initial and final hypersurfaces ${}^3\mathcal{M}(t_\mp)$ and the "side" boundary $[t_-,t_+] \times \partial\,{}^3\mathcal{M}$. Using the value of the signature parameter $\epsilon = -1$ and the change of integration variables from x^μ to (t, x^a) we obtain

$$\begin{aligned} \int d^4x\, g^{1/2} R = &\int_{t_-}^{t_+} dt \int\limits_{{}^3\mathcal{M}(t)} d^3x\, N\gamma^{1/2}\left(K_{ab}^2 - K^2 + {}^3R\right) \\ &-2 \int\limits_{{}^3\mathcal{M}(t)} d^3x\, \gamma^{1/2} K\,\Big|_{t_-}^{t_+} + \int_{t_-}^{t_+} dt \int\limits_{\partial\,{}^3\mathcal{M}(t)} d^2\sigma_a (\ldots)^a. \end{aligned} \tag{4.2}$$

Here we used the relations between the integration measures in the two coordinate systems and the Gauss theorem for the total derivative term, which comprise the material of the next problem.

Problem 4.1. Derive the relation between the integration measures in the original spacetime coordinates x^μ and the coordinates of (3+1)-foliated spacetime (t, x^a)

$$d^4x\, g^{1/2} = dt\, d^3x\, N\, \gamma^{1/2}, \quad \gamma^{1/2} = \big(\det \gamma_{ab}\big)^{1/2}.$$

Check the covariant form of the Gauss theorem

$$\int\limits_{\mathcal{M}^4} d^4x\, g^{1/2} \nabla_\mu \phi^\mu = \int\limits_{\partial\mathcal{M}^4} d^3x\, \gamma^{1/2} \phi_\perp, \quad \phi_\perp = \epsilon\, n_\mu \phi^\mu,$$

where n_μ is an outward pointing normal to the spacetime boundary $\partial\mathcal{M}^4$. **Hint:** Use the Stokes theorem in terms of exterior forms, $\int_{\mathcal{M}^4} d\omega^{(3)} = \int_{\partial\mathcal{M}^4} \omega^{(3)}$, applied to the 3-form $\omega^{(3)} = \phi^\mu \varepsilon_{\mu\alpha\beta\gamma} dx^\alpha \bigwedge dx^\beta \bigwedge dx^\gamma$.

At the bottom and top hypersurfaces ${}^3\mathcal{M}(t_\mp)$ the vector $\phi^\mu = 2\epsilon(n^\nu\nabla_\nu n^\mu - n^\mu\nabla_\nu n^\nu)$, $\phi_\perp = -2\epsilon(\nabla_\nu n^\nu) = -2K$, gives the contribution of the surface term with the trace of extrinsic curvature, whereas at the side boundary it gives the last term of (4.2) which we do not specify exactly because it explicitly depends on its own normal vector different from n_μ. Below we will specify this term for the case of asymptotically flat spacetime.

Now we apply the above formalism for the conversion of the GR action to the canonical form. The Lagrangian action consists of the volume (Einstein-Hilbert) and surface (Gibbons-Hawking) terms and reads

$$S_{\rm G}[\,g_{\mu\nu}\,] = \underbrace{\int\limits_{\mathcal{M}^4} d^4x\, g^{1/2}\,(R - 2\Lambda)}_{\text{EH action}} \underbrace{-2\int\limits_{\partial\mathcal{M}^4} d^3x\,\sqrt{\boldsymbol{\gamma}}\,\boldsymbol{\epsilon}\,\boldsymbol{K}}_{\text{GH action}}\,. \tag{4.3}$$

First of all, to simplify notations, we will work in this section in the system of units $c^4/16\pi G = M_P^2/2 = 1$, which explains the normalization of the volume term. The surface term in the covariant form was suggested by Gibbons and Hawking and represents the surface integral over *entire* boundary of the spacetime domain (4.1) built in terms of the induced metric $\boldsymbol{\gamma}$ and the extrinsic curvature $\boldsymbol{K}$ of this boundary. On top and bottom spacelike surfaces ${}^3\mathcal{M}(t_\pm)$ they obviously coincide respectively with γ_{ab} and K_{ab} and $\boldsymbol{\epsilon} = -1$, whereas on the side boundary $[t_-, t_+]\times\partial^3\mathcal{M}$ they are given by its own induced 3-metric, the normal vector and $\boldsymbol{\epsilon} = 1$. The role of the Gibbons-Hawking term is to guarantee consistency of the variational procedure for the action under the Dirichlet boundary conditions for the induced 3-metric on the total boundary of ${}^4\mathcal{M}$. Simply this means that integration by parts in the spacetime volume allows one to get rid of the second-order derivatives transversal to the boundary and simultaneously avoid such derivatives of the induced metric on the boundary (if any surface terms survive at all).

Such a cancellation of time derivatives takes place when we use (4.2) in (4.3) and thus obtain what is called Arnowitt-Deser-Misner action

$$S_{\rm ADM}[\,\gamma_{ab}, N, N^a\,] = \int_{t_-}^{t_+} dt\,\left\{\int d^3x\,\gamma^{1/2}N\left(K_{ab}^2 - K^2 + {}^{(3)}R - 2\Lambda\right) + \int_{\partial^3\mathcal{M}} d^2\sigma_a\,(\ldots)^a\right\}, \tag{4.4}$$

where K-terms get completely cancelled at ${}^3\mathcal{M}(t_\pm)$ and the ADM Lagrangian involves only first order time derivatives of γ_{ab} (cf. Eq.(3.27) for K_{ab} in terms of $\dot\gamma_{ab}$). At the side boundary a similar cancellation would also take place in the foliation of spacetime by timelike surfaces "parallel" to this boundary, but in the t-foliation this is not explicit – the remaining surface integral $\int dt \int d^2\sigma_a\,(\ldots)^a$ still survives in order to cancel by integration by parts the second order spatial derivatives contained in $N\,{}^{(3)}R$ spacetime integrand. The remnant of this cancellation we will consider later in the case of asymptotically flat spacetime, where it will form the expression for the ADM energy of the gravitational system.

The ADM variables – spatial metric and lapse and shift variables – are most easily related to the original spacetime metric in a special foliation when the time t is identified with x^0 and the intrinsic hypersurface coordinates are identified with spatial coordinates $x^0 = t$, $x^i = x^a$ and $e^\mu_a = \delta^\mu_i$. This relation is given by the solution of the following problem.

Problem 4.2. Prove that in this foliation

$$\begin{aligned}
&\gamma_{ij} = g_{ij}, \quad N_i = g_{0i}, \quad N^i = \gamma^{ij}N_j, \quad N = (-g^{00})^{-1/2},\\
&g^{i0} = \frac{N^i}{N^2}, \quad g^{ij} = \gamma^{ij} - \frac{N^iN^j}{N^2}, \quad g_{00} = N_iN^i - N^2, \quad n^\mu = \left(\frac{1}{N}, -\frac{N^i}{N}\right), \quad n_\mu = -\delta^0_\mu N,
\end{aligned} \tag{4.5}$$

and

$$K_{ij} = n_\mu\nabla_i e^\mu_j = -N\,{}^{(4)}\Gamma^0_{ij} = \frac{1}{2N}\,(D_iN_j + D_jN_i - \dot\gamma_{ij}) \tag{4.6}$$

4.2 Canonical action and Hamiltonian and momenta constraints

The ADM action

$$S_{\rm ADM}[\,\gamma_{ij}, N, N^i\,] = \int_{t_-}^{t_+} dt\,L_{\rm ADM}(\,\gamma_{ij}, \dot\gamma_{ij}, N, N^i\,) \tag{4.7}$$

has the Lagrangian, given by the expression in the curly brackets of (4.4). It contains time derivatives of only the spatial metric coefficients, so that the transition to canonical formalism runs via the Legendre transform with respect to $\dot{\gamma}_{ij}$. We get canonical momenta conjugated to $\gamma_{ij}(\mathbf{x})$,

$$\pi^{ij} = \frac{\delta L_{\rm ADM}}{\delta \dot{\gamma}_{ij}} = -G^{ij,kl} K_{kl}, \tag{4.8}$$

$$G^{ij,kl} \equiv \frac{1}{2}\gamma^{1/2}\left(\gamma^{ik}\gamma^{jl} + \gamma^{il}\gamma^{jk} - 2\gamma^{ij}\gamma^{kl}\right), \tag{4.9}$$

where $G^{ij,kl}$ is the so-called DeWitt metric, and its inverse $G_{ij,kl}$, $G^{ij,kl}G_{kl,mn} = \delta^{ij}_{mn}$, allows one to express $\dot{\gamma}_{ij}$ as functions of momenta π^{ij} and other variables, $\dot{\gamma}_{ij} = \dot{\gamma}_{ij}(\pi^{ij})$,

$$K_{ij} \equiv \frac{1}{2N}\big(\mathscr{D}_i N_j + \mathscr{D}_j N_i - \dot{\gamma}_{ij}\big) = -G_{ij,kl}\pi^{kl}, \quad G_{ij,kl} = \frac{1}{2\gamma^{1/2}}\left(2\gamma_{i(k}\gamma_{jl)} - \gamma_{ij}\gamma_{kl}\right). \tag{4.10}$$

The total Hamiltonian becomes after integration by parts the following expression linear in lapse and shift functions,

$$\left[\int d^3x\, \pi^{ij}\dot{\gamma}_{ij} - L_{\rm ADM}\right]_{\dot{\gamma}_{ij}=\dot{\gamma}_{ij}(\pi^{kl})} = \int_{^3\mathcal{M}} d^3x\,\big(NH_\perp + N^i H_i\big) + \int_{\partial^3\mathcal{M}} d^2\sigma_a\,(\ldots)^a\,, \tag{4.11}$$

playing the role of Lagrange multipliers of the Hamiltonian $H_\perp$ and mpmenta H_i constraint functions,

$$H_\perp = G_{ij,kl}\pi^{ij}\pi^{kl} - \gamma^{1/2}\big({}^{(3)}\!R - 2\Lambda\big), \tag{4.12}$$

$$H_i = -2\gamma_{ij}\mathscr{D}_k\pi^{kj}. \tag{4.13}$$

Thus the canonical ADM action takes the form

$$S_{\rm ADM}[\,\gamma_{ij}, \pi^{ij}, N, N^i\,] = \int_{t_-}^{t_+}\left\{\int_{^3\mathcal{M}} d^3x\,\big(\pi^{ij}\dot{\gamma}_{ij} - NH_\perp - N^i H_i\big) - H_0\right\}, \tag{4.14}$$

$$H_0 = \int_{\partial^3\mathcal{M}} d^2\sigma_a\,(\ldots)^a\,, \tag{4.15}$$

where the part of the total Hamiltonian different from constraints reduces to the surface term denoted by H_0.

To simplify the formalism we again use DeWitt notations and condense spatial coordinates into the gauge condensed index $\mu \mapsto (\perp \mathbf{x}; i\mathbf{x})$, $H_\mu \mapsto (H_\perp(\mathbf{x}), H_i(\mathbf{x}))$, $N^\mu = \big(N^\perp(\mathbf{x}), N^i(\mathbf{x})\big)$, so that the constraint part of the Hamiltonian takes the form of a simple contraction of indices

$$N^\mu H_\mu \equiv \int d^3x\,\big(N^\perp(\mathbf{x})\,H_\perp(\mathbf{x}) + N^i(\mathbf{x})\,H_i(\mathbf{x})\big) \tag{4.16}$$

Then equations of motion for the phase space variables read

$$\begin{bmatrix}\dot{\gamma}_{ij}\\ \dot{\pi}_{ij}\end{bmatrix} = \left\{\begin{bmatrix}\gamma_{ij}\\ \pi_{ij}\end{bmatrix}, H_\mu\right\} N^\mu \tag{4.17}$$

in terms of the Poisson brackets

$$\{A, B\} = \int d^3x \frac{\delta A}{\delta\gamma_{ij}(\mathbf{x})}\frac{\delta B}{\delta\pi^{ij}(\mathbf{x})} - (A \leftrightarrow B)\,. \tag{4.18}$$

Note that H_0 being a surface term does not explicitly contribute to the equations of motion, but as one can check this term provides consistency of the functional variational procedure of deriving these equations in the course of integrations by parts. The equation for $\dot{\gamma}_{ij}$ relating it to π^{ij} is obviously equivalent to the expression for the extrinsic curvature

$$\dot{\gamma}_{ij} = 2N\,G_{ij,kl}\pi^{kl} + 2\mathscr{D}_{(i}N_{j)} = 2\mathscr{D}_{(i}N_{j)} - 2NK_{ij}, \tag{4.19}$$

whereas the equation for $\dot{\pi}^{ij}$ after the substitution of π^{ij} in terms of $\dot{\gamma}_{ij}$ becomes a dynamical equation of second order in time derivatives.

On the contrary the variational equations of motion for the Lagrange multipliers yield the nondynamical constraints on phase space variables

$$\frac{\delta S_{\rm ADM}}{\delta N^\mu} = -H_\mu = 0. \tag{4.20}$$

With the substitution of the Lagrangian values of the canonical momenta these constraints become the equations containing at most first order time derivatives of γ_{ij} and coincide with the following projections of the left hand sides of Einstein equations

$$\begin{aligned} H_\perp\Big|_{\pi^{ij}=-\gamma^{1/2}(K^{ij}-\gamma^{ij}K)} &= -2\gamma^{1/2}\left(G_{\perp\perp} - \Lambda\right) \\ H_i\Big|_{\pi^{ij}=-\gamma^{1/2}(K^{ij}-\gamma^{ij}K)} &= 2\gamma^{1/2} G_{\perp i}. \end{aligned} \tag{4.21}$$

The Poisson bracket algebra of the Hamiltonian and momenta constraints turns out to be closed just like the algebra of constraints in the Yang-Mills theory – the right hand sides of these relations are linear in constraint functions,

$$\begin{aligned} \{H_i(\mathbf{x}), H_j(\mathbf{x}')\} &= H_j(\mathbf{x})\partial_i\delta(\mathbf{x},\mathbf{x}') - (i\mathbf{x} \leftrightarrow j\mathbf{x}') \\ \{H_\perp(\mathbf{x}), H_i(\mathbf{x}')\} &= -H_\perp(\mathbf{x}')\partial_i\delta(\mathbf{x},\mathbf{x}') \\ \{H_\perp(\mathbf{x}), H_\perp(\mathbf{x}')\} &= \gamma^{ij}(\mathbf{x})H_i(\mathbf{x})\partial_j\delta(\mathbf{x},\mathbf{x}') - (\mathbf{x} \leftrightarrow \mathbf{x}')\,. \end{aligned} \tag{4.22}$$

These relations can be relatively easily proven by considering the constraints smeared by contraction with the test functions, $H_i(\mathbf{x}) \to H_{\vec{\xi}} \equiv \int d^3x\, H_i(\mathbf{x})\xi^i(\mathbf{x})$, $H_\perp(\mathbf{x}) \to H_\varphi \equiv \int d^3x\, H_\perp(\mathbf{x})\varphi(\mathbf{x})$, and calculating their Poisson brackets. Again, the constraint algebra very compactly reads in terms of DeWitt notations

$$\{H_\mu, H_\nu\} = \mathcal{U}^\lambda_{\mu\nu} H_\lambda, \tag{4.23}$$

where the structure *functions* (note that $\mathcal{U}^{i\mathbf{z}}_{\perp\mathbf{x}\,\perp\mathbf{x}'}$ depends on γ^{ij}) can be read off the relations

$$\mathcal{U}^{\perp\mathbf{x}}_{\mu\nu}\mathcal{F}^\mu_1\mathcal{F}^\nu_2 = -\mathcal{F}^i_1(\mathbf{x})\,\partial_i\mathcal{F}^\perp_2(\mathbf{x}) + \mathcal{F}^i_2(\mathbf{x})\,\partial_i\mathcal{F}^\perp_1(\mathbf{x}), \tag{4.24}$$

$$\mathcal{U}^{i\mathbf{x}}_{\mu\nu}\mathcal{F}^\mu_1\mathcal{F}^\nu_2 = -\mathcal{F}^j_1(\mathbf{x})\,\partial_j\mathcal{F}^i_2(\mathbf{x}) + \mathcal{F}^j_2(\mathbf{x})\,\partial_j\mathcal{F}^i_1(\mathbf{x}) + \mathcal{F}^\perp_1(\mathbf{x})\,\mathscr{D}^i\mathcal{F}^\perp_2(\mathbf{x}) - \mathcal{F}^\perp_2(\mathbf{x})\,\mathscr{D}^i\mathcal{F}^\perp_1(\mathbf{x}) \tag{4.25}$$

In complete analogy with the Yang-Mills theory let us define gauge transformations of the canonical action, $\delta^{\mathcal{F}} S_{\rm ADM}[\,\gamma_{ij}, \pi^{ij}, N, N^i\,] = 0$,

$$\delta^{\mathcal{F}}\begin{bmatrix}\gamma_{ij}\\ \pi^{ij}\end{bmatrix} = \left\{\begin{bmatrix}\gamma_{ij}\\ \pi^{ij}\end{bmatrix}, H_\mu\right\}\mathcal{F}^\mu, \quad \delta^{\mathcal{F}} N^\mu = \dot{\mathcal{F}}^\mu - \mathcal{U}^\mu_{\alpha\beta}N^\alpha\mathcal{F}^\beta, \tag{4.26}$$

and show that these transformations are compatible with diffeomorphism invariance of the Lagrangian action $\Delta^f S_{\rm G}[\,g_{\alpha\beta}\,] = 0$, $\Delta^f g_{\alpha\beta} = -\nabla_\alpha f_\beta - \nabla_\beta f_\alpha$, under a proper relation between the gauge transformation parameters $\mathcal{F}$ and f. We have $\delta^{\mathcal{F}}\gamma_{ij} = 2G_{ij,kl}\pi^{kl}\mathcal{F}^\perp + 2\mathscr{D}_{(i}\,\mathcal{F}_{j)}$, so that at the Lagrangian values of the momenta

$$\delta^{\mathcal{F}}\gamma_{ij}\Big|_{\pi^{ij}=-G^{ij,kl}K_{kl}} = 2\mathscr{D}_{(i}\,\mathcal{F}_{j)} - 2K_{ij}\mathcal{F}^\perp. \tag{4.27}$$

On the other hand, the diffeomorphism transformation of the spatial metric with fixed embedding functions $e^\alpha(\mathbf{x}, t)$ (note that these functions are not field variables and play inert role under spacetime diffeomorphisms) reads

$$\begin{aligned} \Delta^f\gamma_{ij} &= e^\alpha_i e^\beta_j \Delta^f g_{\alpha\beta} = -2\nabla_{(i}f_{j)} = -2e^\alpha_{(i}\nabla_\alpha\big(e^\beta_{j)}f_\beta\big) + 2\big(\nabla_{(i}e^\beta_{j)}\big)f_\beta \\ &= -2e^\alpha_{(i}\partial_\alpha f_{j)} + 2\gamma^l_{ij}f_l + 2K_{ij}f^\perp = -2\mathscr{D}_{(i}\,f_{j)} + 2K_{ij}f^\perp, \end{aligned} \tag{4.28}$$

where we first used the Leibnitz rule for covariant derivatives and then the Gauss-Weingarten equation (3.19) for $\nabla_{(i}e^\beta_{j)}$. Therefore,

$$\delta^{\mathcal{F}}\gamma_{ij} = \Delta^f\gamma_{ij}, \tag{4.29}$$

provided $\mathcal{F}^\perp = -f_\perp$ and $\mathcal{F}^i = -f^i$. Thus $\mathcal{F}^\perp$ and $\mathcal{F}^i$ up to a sign are just normal and tangential projections of the diffeomorphism vector field f^μ onto a normal basis,

$$f^\mu = -n^\mu \mathcal{F}^\perp - e^\mu_i \mathcal{F}^i. \tag{4.30}$$

In the same way for the diffeomorphism of the lapse function we have a chain of relations

$$\begin{aligned}\Delta^f N = -\Delta^f n_\alpha \frac{\partial e^\alpha}{\partial t} = \nabla_\perp f_\perp N = N n^\mu n^\nu \nabla_\mu f_\nu = N n^\mu \nabla_\mu (n^\nu f_\nu) - N(n^\mu \nabla_\mu n^\nu) f_\nu \\ = -(\dot{e}^\mu \partial_\mu - N^a \partial_a) f_\perp - f^a \partial_a N = -\dot{f}_\perp + N^a \partial_a f_\perp - f^a \partial_a N = \dot{\mathcal{F}}^\perp - \mathcal{U}^\perp_{\mu\nu} N^\mu \mathcal{F}^\nu = \delta^{\mathcal{F}} N.\end{aligned} \tag{4.31}$$

Here, from the solution of the Problem 3.2, it was used that $n^\mu \nabla_\mu n^\nu = -\epsilon e^\nu_a \mathscr{D}^a N/N$, the diffeomorphism of the normal vector follows from the solution of the Problem 4.3 below and the last two equalities are based on the relation (4.24). Similar proof holds for the shift functions $\Delta^f N^i = \delta^{\mathcal{F}} N^i$.

Problem 4.3. Derive the variation of the normal vector $\delta_g n_\alpha$ under an arbitrary infinitesimal variation of the metric $\delta g_{\mu\nu}$ and show that the diffeomorphism transformation of this vector equals $\Delta^f n_\alpha = -\epsilon n_\alpha n^\mu n^\nu \nabla_\mu f_\nu$.

Lecture 5. Generic systems subject to first class constraints

- Gauge invariance in generic system subject to first class constraints
- Canonical gauge fixing procedure
- Reduction to the physical sector
- Integration measure on the physical phase space
- Physical sector in time-dependent gauges: relativistic particle and linearized GR

Examples of canonical formalism for gauge invariant models considered above – those of relativitic particle, YM theory and GR – can be universally described within condensed DeWitt notations along the following lines.

5.1 Gauge invariance in generic system subject to first class constraints

Consider a general dynamical system of variables (q^i, λ^μ) labelled by condensed indices i and μ have the formal range

$$i = 1, 2, ...n, \quad n = \text{range}\, i, \tag{5.1}$$

$$\mu = 1, 2, ...m, \quad m = \text{range}\, \mu, \tag{5.2}$$

with some $n > m$ which in local field models can be formally infinite. Let the Lagrangian of the model $L = L(q, \dot{q}, \lambda)$ be independent of the time derivatives of the variables λ^μ and such that it gives rise to the canonical formalism of the theory with the action

$$S[\,q, p, \lambda\,] = \int_{t_-}^{t_+} dt\, \big(p_i \dot{q}^i - H_0(q,p) - \lambda^\mu T_\mu(q,p)\big)\,, \tag{5.3}$$

where $H_0(q,p)$ is the Hamiltonian, λ^μ play the role of Lagrange multipliers of constraints $T_\mu(q,p)$ which satisfy the commutation relations

$$\{T_\mu, T_\nu\} = \mathcal{U}^\lambda_{\mu\nu} T_\lambda, \quad \{H_0, T_\mu\} = \mathcal{V}^\nu_\mu T_\nu \tag{5.4}$$

with some coefficient functions $\mathcal{U}^\lambda_{\mu\nu}$ and $\mathcal{V}^\nu_\mu$ on phase space of the model. Within the terminology of Dirac constrained systems the constraints having this property are called *first class* constraints. Let us show now that this canonical action is invariant,

$$\delta^{\mathcal{F}} S[\,q, p, \lambda\,] = 0, \tag{5.5}$$

under the local gauge transformation with the parameters $\mathcal{F}^\nu = \mathcal{F}^\nu(t)$ arbitrarily depending on time and vanishing at $t_\pm$,

$$\begin{aligned}&\delta^{\mathcal{F}} q^i = \{q^i, T_\mu\}\,\mathcal{F}^\mu, \quad \delta^{\mathcal{F}} p_i = \{p_i, T_\mu\}\,\mathcal{F}^\mu\\ &\delta^{\mathcal{F}}\lambda^\mu = \dot{\mathcal{F}}^\mu - \mathcal{U}^\mu_{\nu\lambda}\lambda^\nu\mathcal{F}^\lambda - \mathcal{V}^\mu_\nu\mathcal{F}^\nu.\end{aligned} \tag{5.6}$$

As in examples above the first two transformations are canonical with the generating function $F = T_\mu\mathcal{F}^\mu$. For time independent $\mathcal{F}^\mu$, $\dot{\mathcal{F}}^\mu = 0$, these transformations shift the symplectic term by the total derivative term, as it should be for a canonical transformation

$$\begin{aligned}\delta^{\mathcal{F}}(pdq) &= \{p, F\}\,dq + pd\,\{q, F\} = -\frac{\partial F}{\partial q}dq + pd\left(\frac{\partial F}{\partial p}\right) = -\frac{\partial F}{\partial q}dq - \frac{\partial F}{\partial p}dp + d\left(p\frac{\partial F}{\partial p}\right)\\ &= d\left(p\frac{\partial F}{\partial p} - F\right)\end{aligned} \tag{5.7}$$

For time dependent $\mathcal{F}^\mu(t)$ this transformation however brings into the integrand extra term obtained from $-\dot{T}_\mu\mathcal{F}^\mu = -d(T_\mu\mathcal{F}^\mu)/dt + T_\mu\dot{\mathcal{F}}^\mu$. We have

$$\begin{aligned}&\delta^{\mathcal{F}}\int_{t_-}^{t_+} dt\,\left(p_i\dot{q}^i - H_0 - \lambda^\mu T_\mu\right)\\ &\quad= \int_{t_-}^{t_+} dt\left[-\frac{\partial T_\mu}{\partial q^i}\mathcal{F}^\mu\dot{q}^i + p_i\frac{d}{dt}\left(\frac{\partial T_\mu}{\partial p_i}\mathcal{F}^\mu\right) - \{H_0, T_\mu\}\,\mathcal{F}^a - (\delta^{\mathcal{F}}\lambda^\mu)\,T_\mu - \lambda^\mu\{T_\mu, T_\nu\}\,\mathcal{F}^\nu\right]\\ &\quad= p\frac{\partial T_\mu}{\partial p}\mathcal{F}^\mu\Big|_{t_-}^{t_+} + \int_{t_-}^{t_+} dt\,\left[-\dot{T}_\mu\,\mathcal{F}^\mu - \mathcal{V}^\nu_\mu\,T_\nu\mathcal{F}^\mu - (\delta^{\mathcal{F}}\lambda^\mu)\,T_\mu - \lambda^\mu\,\mathcal{U}^\lambda_{\mu\nu}\,T_\lambda\,\mathcal{F}^\nu\right]\\ &\quad= \left(p_i\frac{\partial T_\mu}{\partial p_i} - T_\mu\right)\mathcal{F}^\mu\Big|_{t_-}^{t_+} + \int_{t_-}^{t_+} dt\,T_\mu\left(\dot{\mathcal{F}}^\mu - V^\mu_\nu\mathcal{F}^\nu - \mathcal{U}^\mu_{\nu\lambda}\lambda^\nu\mathcal{F}^\lambda - \delta^{\mathcal{F}}\lambda^\mu\right) = 0,\end{aligned} \tag{5.8}$$

where we took into account that $\mathcal{F}^\mu(t_\pm) = 0$. This derivation explains the origin of the structure constants (or functions) term in the transformation (2.13) for YM theory and GR. The second set of structure functions $\mathcal{V}^\mu_\nu$ in the Poisson brackets commutator of constraints with the Hamiltonian H_0 in all the examples above turns out to be absent? but generically it is nonzero and provides extra contribution to $\delta^{\mathcal{F}}\lambda$.

5.2 Canonical gauge fixing procedure

As in all models considered above the variational equations for the canonical action lead to Hamiltonian (evolutionary) equations for phase space variables and the set of constraints imposed on the latter at any moment of time

$$\frac{\delta S}{\delta\begin{pmatrix} q\\ p\end{pmatrix}} = 0 \;\rightarrow\; \frac{d}{dt}\begin{pmatrix} q\\ p\end{pmatrix} = \left\{\begin{pmatrix} q\\ p\end{pmatrix}, H_0\right\} + \left\{\begin{pmatrix} q\\ p\end{pmatrix}, T_\mu\right\}\lambda^\mu, \tag{5.9}$$

$$\frac{\delta S}{\delta\lambda^\mu} = 0 \;\rightarrow\; T_\mu = 0. \tag{5.10}$$

The Lagrange multipliers λ^μ stay completely arbitrary, which obviously corresponds to the fact that the action is invariant under gauge transformations with the parameters $\mathcal{F}^\mu$ whose number (the range of index μ) coincides with the number of λ^μ.

The consistency of the constraints $T_\mu = 0$ with canonical equations for (q^i, p_i) is fully satisfied because their conservation in time is enforced due to the Poisson bracket algebra of constraints

$$\frac{dT_\mu}{dt}\Big|_{T=0} = \{T_\mu, H_0\} + \{T_\mu, T_\nu\}\,\lambda^\nu = \left(-\mathcal{V}^\lambda_\mu + \mathcal{U}^\lambda_{\mu\nu}\lambda^\nu\right)T_\lambda\Big|_{T=0} = 0. \tag{5.11}$$

Thus it is enough to enforce the constraints at the initial moment of time and they will hold all the time.

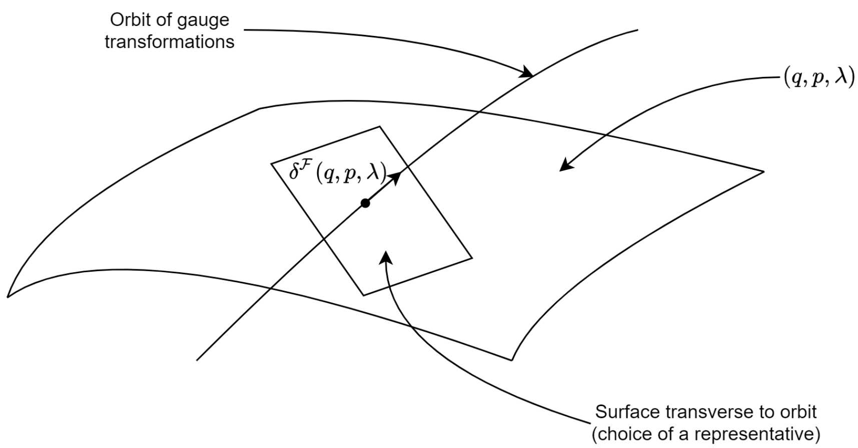

Figure 5: Selection of the representative of the class of physically equivalent configurations as a choice of the single crossing point of the orbit of the gauge group and the surface of gauge conditions.

Arbitrariness of the solution of equations of motion parameterized by m generic functions of time $\mathcal{F}^\mu(t)$ implies that the physical state of the system is characterized not by a concrete history in the configuration space of $\big(q^i(t), p_i(t), \lambda^\mu(t)\big)$ but by the class of equivalence of histories joined by the transformations (5.6). To label different physical states we, therefore, have to select a single representative of every equivalence class, so that the full set of these representatives will form a *physical sector* of the theory. The procedure of selecting this representative is shown on Fig.5. The transformations $\delta^{\mathcal{F}}(q,p,\lambda)$ form the m-dimensional surface in $(2n+m)$-dimensional space of (q,p,λ) – the *orbit* of the gauge group. The physical representative can be chosen as a single crossing point of this orbit with the transversal surface of the complementary dimensionality $(2n+m)-m = 2n$. This transversal surface can be chosen by imposing m *gauge conditions* $\chi^\mu\,(q,p,\lambda) = 0$ on $(2n+m)$ variables of the full configuration space, and the requirement that this crossing point is at least locally *unique* is that the transformation $\delta^{\mathcal{F}}(q,p,\lambda)$ with any nonzero $\mathcal{F}^\mu$ will shift the point from the surface of gauge conditions $\chi^\mu((q,p,\lambda) = 0$. This admissibility condition for the choice of gauge can be written down as the requirement that the equation $\delta^{\mathcal{F}}\chi^\mu\,(q,p,\lambda) = 0$ holds if and only if all $\mathcal{F}^\mu$ are identically zero, or the requirement that the following equation has only identically vanishing solution for $\mathcal{F}^\mu(t)$

$$\delta^{\mathcal{F}}\chi^\mu\,(q,p,N) = \{\chi^\mu, T_\nu\}\,\mathcal{F}^\nu + \frac{\partial\chi^\mu}{\partial\lambda^\nu}\left(\frac{d}{dt}\mathcal{F}^\nu + ...\right) = 0 \;\Leftrightarrow\; \mathcal{F}^\mu = 0. \qquad (5.12)$$

For gauge functions $\chi^\mu\,(q,p,\lambda)$ depending on Lagrange multipliers this is impossible because the system of first order differential equations in time derivatives for generic initial conditions always has a nontrivial solution. This means that gauge conditions which uniquely select the physical sector of the theory should be imposed only on phase space variables, $\chi^\mu\,(q,p) = 0$, and their admissibility requirement reduces to the invertibility of a special *ultralocal* in time matrix J^μ_ν,

$$J^\mu_\nu = \{\chi^\mu, T_\nu\}\,. \qquad (5.13)$$

Indeed,

$$\chi^\mu = 0 \;\Rightarrow\; \delta^{\mathcal{F}}\chi^\mu = \{\chi^\mu, T_\nu\}\,\mathcal{F}^\nu = 0 \;\Leftrightarrow\; \mathcal{F}^\nu = 0 \;\Rightarrow\; \det\{\chi^\mu, T_\nu\} \neq 0. \qquad (5.14)$$

Such gauge fixing procedure simultaneously fixes the choice of Lagrange multipliers. This choice uniquely follows from the requirement of gauge conditions conservation in time

$$\frac{d}{dt}\chi^\mu = \{\chi^\mu, H_0\} + \{\chi^\mu, T_\nu\}\,\lambda^\nu = 0, \qquad (5.15)$$

and the invertibility of (5.13),

$$\lambda^\nu = -J^{-1}{}^\nu_\mu\,\{\chi^\mu, H_0\}\,. \qquad (5.16)$$

With these values of Lagrange multipliers equations of motion in phase space can be rewritten in terms of the so-called Dirac bracket. To define it introduce the full set of constraints arising after gauge fixing $C_a = (T_\mu, \chi^\nu)$.

In contrast to T_μ they form the set of *second class* constraints, because their Poisson brackets on the subspace of first class constraints $T_\mu = 0$ form the matrix $D_{ab} \equiv \{C_a, C_b\}$ which is nondegenerate and invertible

$$D_{ab}\Big|_{T=0} = \begin{bmatrix} \{T,T\} & \{T,\chi\} \\ \{\chi,T\} & \{\chi,\chi\} \end{bmatrix}\Big|_{T=0} = \begin{bmatrix} 0 & -J \\ J & \{\chi,\chi\} \end{bmatrix}. \tag{5.17}$$

Gauge conditions on equal footing with the first class constraints can be included into the canonical action with the full set of Lagrange multipliers $\Lambda^a = (\lambda^\mu, \pi_\nu)$, $S[\,q,p,\Lambda\,] = \int dt\,\big(p\dot q - H_0 - \Lambda^a C_a\big)$. Similarly to (5.16) the constraints conservation leads to $\Lambda^a = -D^{-1\,ab}\{C_b, H_0\}$, and the canonical equations of motion take the form

$$\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \left\{\begin{pmatrix} q \\ p \end{pmatrix}, H_0\right\}_{\rm D}, \tag{5.18}$$

where the Dirac bracket on phase space is defined as

$$\{A,B\}_{\rm D} = \{A,B\} - \{A,C_a\}\,D^{-1\,ab}\{C_b,B\}. \tag{5.19}$$

Problem 5.1. Prove that for any observable ϕ on phase space

$$\dot\phi = \{\phi, H_0\}_D\,\big|_{C_a=0} = \{\phi, H_0\} - \{\phi, T_\mu\}\,J^{-1\mu}{}_\nu\{\chi^\nu, H_0\}.$$

5.3 Reduction to the physical sector

Thus on equations of motion after imposing canonical gauge conditions Lagrange multipliers λ become functions of (q,p). Do q and p remain dynamically independent? The answer is of course "no", because these $2n$ variables are subject to the full set of $2m$ first class $T_\mu(q,p) = 0$ and second class $\chi^\mu(q,p) = 0$ constraints. Therefore, there is only $2(n-m)$ independent degrees of freedom which can somehow be chosen to parameterize the phase space of the physical sector of the theory. Let us denote them as the coordinates ξ^A and the canonically conjugated momenta π_A, labelled by a condensed index A,

$$A = 1, 2, ...n-m, \tag{5.20}$$

with the range $(n-m)$ – the number of physical *degrees of freedom*.

Constructively, these variables can be most easily built within the class of *coordinate* gauge conditions,

$$\chi^\mu(q) = 0, \tag{5.21}$$

imposed only on coordinates q^i of the full phase space. These m gauge conditions determine the $(n-m)$-dimensional subspace Σ in the n-dimensional coordinate space. The embedding of this subspace can be described by embedding functions of the internal coordinates on Σ,

$$\Sigma: \quad q^i = e^i(\xi^A), \tag{5.22}$$

which identically satisfy the equation

$$\chi^\mu\big(e^i(\xi^A)\big) = 0. \tag{5.23}$$

The geometry of embedding the physical subspace Σ into the q-space is very similar to the geometry of (3+1)-foliation of spacetime considered above, except that Σ is not a hypersurface and has a nontrivial codimension,

$$\text{codim}\,\Sigma = m, \quad \dim\Sigma = n-m. \tag{5.24}$$

Correspondingly it has not one but m covariant normal vectors given by the gradient of the constraints functions,

$$\chi^\mu_i = \frac{\partial\chi^\mu}{\partial q^i}, \tag{5.25}$$

and labelled by the index μ. Differentiation of the identity (5.23) confirms that they are indeed normal to the vectors e^i_A tangential to Σ,

$$e^i_A \equiv \frac{\partial e^i(\xi)}{\partial \xi^A}, \quad e^i_A \chi^\mu_i = 0. \tag{5.26}$$

Thus, the set (χ^μ_i, e^i_a) forms the complete basis of covariant normal and contravariant tangential vectors.

To proceed further one should introduce the metric G_{ik} on the q-manifold with $G^{ik} = (G_{ik})^{-1}$.[1] Then one can define the induced metric G_{AB} on Σ and get the possibility of raising and lowering the indices i and A,

$$G_{AB} \equiv G_{ik} e^i_A e^k_B, \quad G^{AB} = (G_{AB})^{-1}, \quad e^A_i = G^{AB} e^k_B G_{ki}, \quad \chi^{\mu i} = G^{ik} \chi^\mu_k. \tag{5.27}$$

Similarly one introduces the analogue of the contravariant metric in gauge directions orthogonal to Σ,

$$G^{\mu\nu} \equiv \chi^\mu_i G^{ik} \chi^\nu_k, \quad G_{\mu\nu} = (G^{\mu\nu})^{-1}, \tag{5.28}$$

which allows one also to raise and lower the gauge indices. All this gives the possibility to expand every vector or covector in the normal basis of (χ^μ_i, e^i_a), $v_i = v_A e^A_i + v_\mu \chi^\mu_i$, with the projections

$$v_A = e^i_A v_i, \quad v_\mu = G_{\mu\nu} \chi^\nu_i G^{ik} v_k. \tag{5.29}$$

The symplectic term of the canonical action reduced to the physical subspace takes the form

$$\int dt\, p_i \dot{q}^i \Big|_{\chi^\mu = 0} = \int dt\, p_i \frac{\partial e^i(\xi)}{\partial \xi^A}\, \dot{\xi}^A, \tag{5.30}$$

which allows one to identify the tangential projections of the momentum covector p_i with the physical momenta conjugated to ξ^A,

$$\pi_A = e^i_A p_i. \tag{5.31}$$

In order to find the normal projections of p_i one should substitute the decomposition (5.29) for the momentum into the first class constraints

$$T_\mu\big(q^i, \pi_A e^i_A + p_\nu \chi^\nu_i\big) \Big|_{q^i = e^i(\xi)} = 0. \tag{5.32}$$

The necessary condition for local solvability of this equation with respect to p_ν is guaranteed by the nondegeneracy of the matrix (5.13) because

$$\frac{\partial}{\partial p_\nu} T_\mu\big(q^i, \pi_A e^A_i + p_\nu \chi^\nu_i\big) = \frac{\partial T_\mu}{\partial p_i} \chi^\nu_i = \{\chi^\nu, T_\mu\}, \tag{5.33}$$

so that the solution gives a locally unique $p_\mu = P_\mu(\xi, \pi)$ as a function of physical canonical variables, and the full momentum p_i in the physical sector of the theory becomes $p_i(\xi, \pi) = \pi_A e^A_i + P_\mu(\xi, \pi) \chi^\mu_i$. Substituting this together with (5.22) into the original canonical action (5.3) we obtain the canonical action in this sector as a functional of physical phase space variables

$$S_{\rm phys}[\,\xi, \pi\,] = S[\,q, p, \lambda\,] \Big|_{T_\mu = 0,\ \chi^\mu = 0} = \int_{t_-}^{t_+} dt\, \Big(\pi_A \dot{\xi}^A - H_0\big(e(\xi), p(\xi, \pi)\big)\Big). \tag{5.34}$$

5.4 Integration measure on the physical phase space

Below we will need to make the change of functional integration variables from the path integral over physical phase space to the path integral over the original phase space of (q, p). Let us first do this for the coordinates ξ^A and q^i. Introduce in the vicinity of Σ (that is at $\chi^\mu(q) = 0$) the variables $\theta^\mu = \chi^\mu(q)$ which obviously equal

[1] Such a metric always exists in concrete models. In GR this is just the DeWitt metric on the space of 3-metrics γ_{ij}, $G_{ik} \mapsto G^{ij,kl}(\mathbf{x})\delta(\mathbf{x}, \mathbf{y})$, $i \mapsto (ij, \mathbf{x})$, $k \mapsto (kl, \mathbf{y})$, whereas in YM theory with $q^i \mapsto A^a_i(\mathbf{x})$, this is just the Killing metric of the generating YM group, $G_{ik} \mapsto \gamma_{ab} \delta^{ij}(\mathbf{x})\delta(\mathbf{x}, \mathbf{y})$, $i \mapsto (ia, \mathbf{x})$, $k \mapsto (jb, \mathbf{y})$, etc.

zero on Σ, $\theta^\mu\,|_{\,\Sigma=0}$. Then one can write the chain of relations for the measures of integration over Σ and over full q-space, $d^{(n-m)}\xi \equiv \prod_{A=1}^{n-m} d\xi^A$ and $d^{(n)}q \equiv \prod_{i=1}^{n} dq^i$

$$d^{(n-m)}\xi = d^{(n-m)}\xi\, d^{(m)}\theta\delta(\theta) = d^{(n)}q\,\delta(\chi)\left|\frac{\partial(\xi,\theta)}{\partial q}\right|, \tag{5.35}$$

where $\delta(\theta) \equiv \prod_{\mu=1}^{m}\delta(\theta^\mu)$ and, obviously, $\delta(\chi) \equiv \prod_{\mu=1}^{m}\delta(\chi^\mu(q))$. To calculate the Jacobian of transition $(\xi^A,\theta^\mu) \to q^i = e^i(\xi^A,\theta^\mu)$ we use the identities

$$e^i(\xi,0) = e^i(\xi), \quad \chi^\mu\big(e^i(\xi,\theta)\big) \equiv \theta^\mu, \tag{5.36}$$

where $e^i(\xi)$ are the embedding functions (5.22). As a corollary of the second identity we have the relation

$$\chi^\mu_i\,\frac{\partial e^i}{\partial\theta^\nu} = \delta^\mu_\nu, \tag{5.37}$$

so that the decomposition of $\partial e^i/\partial\theta^\nu$ in the normal basis takes the form

$$\frac{\partial e^i}{\partial\theta^\nu} = \chi^i_\nu + e^i_A(...)^A_\nu, \tag{5.38}$$

with some tangential projection $(...)^A_\nu$. Then the Jacobian of transformation equals

$$\left|\frac{\partial(\xi,\theta)}{\partial q}\right|^{-1} = \det\left[\, e^i_A \;\vdots\; \chi^i_\nu + e^i_A(...)^A_\nu \,\right] = \det\left[\, e^i_A \;\vdots\; \chi^i_\nu \,\right], \tag{5.39}$$

where the tangential projection $(...)^A_\nu$ does not contribute because addition of a column to the matrix does not change the value of its determinant. To calculate the last determinant consider the matrix equation which holds in view of the orthogonality property $e^i_A\chi^\mu_i = 0$,

$$\left[\, e^i_A \;\vdots\; \chi^i_\mu \,\right] G_{ik} \begin{bmatrix} e^k_B \\ \chi^k_\nu \end{bmatrix} = \begin{bmatrix} G_{AB} & 0 \\ 0 & G_{\mu\nu} \end{bmatrix} \tag{5.40}$$

Taking the determinant of both sides of this relation we get

$$\left[\, e^i_A \;\vdots\; \chi^i_\mu \,\right] = \frac{1}{\mathcal{M}}, \quad \mathcal{M} \equiv \left(\frac{\det G_{ik}}{\det G_{AB}\,\det G_{\mu\nu}}\right)^{1/2}, \tag{5.41}$$

so that from (5.35) the relation between the integration measures on the q-space and Σ reads

$$d^{(n-m)}\xi = d^{(n)}q\,\delta(\chi)\,\mathcal{M}. \tag{5.42}$$

Note that the metric G_{ik} on the q-space plays auxiliary role, so that the measure factor should be G_{ik}-independent, which can be shown by solving the next problem:

Problem 5.2. Prove G_{ik}-metric independence of the integration measure $\mathcal{M} = (\det G_{ik}/\det G_{AB}\,\det G_{\mu\nu})^{1/2}$

$$\frac{\partial\mathcal{M}}{\partial G_{ik}} = 0$$

and the application of this measure in the case of (3+1)-foliation of spacetime, $N\sqrt{\gamma}\,d^3x = \sqrt{g}\,d^4x\,\delta(x^0-t)$.

The transformation of the integration measure factor on the space of momenta gives in view of $p_i = \pi_A e^A_i + p_\nu\chi^\nu_i$

$$d^{(n)}p = \left|\frac{\partial p_i}{\partial(\pi_A,p_\mu)}\right| d^{(m)}p\,d^{(n-m)}\pi = \det\left[\, e^A_i \;\vdots\; \chi^\mu_i \,\right] d^{(m)}p\,d^{(n-m)}\pi = \mathcal{M}\,d^{(m)}p\,d^{(n-m)}\pi, \tag{5.43}$$

because the matrix $\left[\, e^A_i \;\vdots\; \chi^\mu_i \,\right]$ is obviously inverse to $\left[\, e^i_A \;\vdots\; \chi^i_\mu \,\right]$. Therefore,

$$d^{(n-m)}\pi = d^{(n-m)}\pi \prod_{\mu=1}^{m} dp_\mu\, \delta\big(p_\mu - P_\mu(\xi,\pi)\big)$$
$$= d^{(n)}p \frac{1}{\mathcal{M}}\, \det \frac{\partial T_\mu(q,p)}{\partial p_\nu} \prod_{\mu=1}^{m} \delta(T_\mu) = d^{(n)}p \frac{1}{\mathcal{M}}\, \delta(T)\, \det J^\mu_\nu, \tag{5.44}$$

so that finally we obtain the fundamental relation between the Liouville integration measures on the original full and physical (reduced) phase space in the set of canonical gauge conditions $\chi = \chi^\mu(q,p)$

$$d^{(n-m)}\xi\, d^{(n-m)}\pi = d^{(n)}q\, d^{(n)}p\, \delta(T)\, \delta(\chi)\, \det\{\chi^\mu, T_\nu\}. \tag{5.45}$$

The integration measure is restricted to the subspace of the full set of first-class constraints and gauge fixing conditions and contains a nontrivial factor — the *Faddeev-Popov determinant* which will play important role in the canonical quantization of gauge theories.

5.5 Physical sector in time-dependent gauges: relativistic particle and linearized GR

If we apply the above reduction algorithm to the relativistic particle model or general relativity we immediately run into trouble. Since the Hamiltonian H_0 is identically vanishing, in any canonical gauge of the above type the Lagrangian multipliers $\lambda^\mu \sim \{\chi, H_0\} = 0$. This is obviously contradictory both in relativistic particle case with $\lambda^\mu = N$ and the GR case with $\lambda^\mu = \big(N^\perp(\mathbf{x}), N^i(\mathbf{x})\big)$, because this is geometrically absurd. In GR, in particular, this would mean that the "velociity" N^μ with which the spacial slice is moving in spacetime is vanishing – spacetime (3+1)-foliation degenerates to just one fixed space hypersurface which no longer spans the whole of spacetime.

To circumvent this difficulty we generalize the above reduction procedure to the class of canonical gauges *explicitly* depending on time t,

$$\chi^\mu(q) = 0 \quad\Rightarrow\quad \chi^\mu(q,t) = 0 \quad\Rightarrow\quad q^i = e^i(\xi^A, t), \tag{5.46}$$

so that the physical space embedding functions also become explicitly t-dependent. Correspondingly the condition of conservation of the gauge conditions in time becomes

$$\frac{d}{dt}\chi^\mu(q,t) = \{\chi^\mu, H_0\} + \{\chi^\mu, T_\nu\}\lambda^\nu + \frac{\partial \chi^\mu}{\partial t} = 0 \tag{5.47}$$

and results in nonvanishing Lagrange multipliers even for zero Hamiltonian $H_0 = 0$,

$$\lambda^\mu = -J^{-1}{}^\mu_\nu\{\chi^\mu, H_0\} - J^{-1}{}^\mu_\nu \frac{\partial \chi^\nu}{\partial t} \neq 0. \tag{5.48}$$

The reduction of the canonical action to the physical sector for $H_0 = 0$ goes obviously as follows

$$S[\,q,p,\lambda\,]\,\big|_{T=\chi=0} = \int dt\, p_i \dot q^i\,\big|_{T=\chi=0} = \int dt\, \left(\pi_A \dot\xi^A + p_i \frac{\partial e^i(\xi,t)}{\partial t}\right) = \int dt\, \Big(\pi_A \dot\xi^A - H_{\rm phys}(\xi,\pi,t)\Big), \tag{5.49}$$

the physical Hamiltonian being generated by explicit time derivative of the embedding functions.

For the relativistic particle in a special gauge we have

$$q^i = x^\alpha, \quad p_i = p_\alpha, \quad \alpha = 0,1,...3; \quad T_\mu = T(p) = p^2 + m^2, \tag{5.50}$$
$$\chi(x^\alpha, t) = x^0 - t, \quad J = \{\chi, T\} = -2p_0, \tag{5.51}$$
$$e^i(\xi,t): \quad x^0 = t, \;\; x^i = \xi^i \equiv \mathbf{x}, \quad p_i = \pi_i \equiv \mathbf{p}, \tag{5.52}$$

along with the solution of the constraint $p_0 = \mp\sqrt{\mathbf{p}^2 + m^2}$ and the resulting physical action

$$S[\,\mathbf{x},\mathbf{p}\,] = \int dt\, \big(\mathbf{p}\dot{\mathbf{x}} \mp \sqrt{\mathbf{p}^2 + m^2}\big). \tag{5.53}$$

Double-fold solution corresponds to the degeneration of the Faddeev-Popov operator at $p_0 = 0$ where $J = 0$. This separates positive and negative energy solutions and indicates the presence of Gribov copies – several (two) representatives of the equivalence class of phase space configurations in the used gauge. This problem is being solved only within secondary quantization framework which corresponds to raising quantum mechanics to the level of QFT.

In Einstein GR we have $q^i = \gamma_{ij}(\mathbf{x})$, $p_i = \pi^{ij}(\mathbf{x})$, and condensed indices read

$$i \mapsto (ij, \mathbf{x}), \quad n = 6 \times \infty^3, \tag{5.54}$$

$$\mu \mapsto (\mu, \mathbf{x}), \quad m = 4 \times \infty^3 \tag{5.55}$$

Number of physical degrees of freedom $n - m = 2 \times \infty^3$ – two degrees of freedom per spatial point – formal dimensionality of the physical subspace Σ. This subspace is selected by some m gauge conditions $\chi^\mu(h_{ij}, t) = 0$. For spatially closed cosmology with $H_0 = 0$ the only way to make the model evolving in time with nonvanishing lapse and shift functions is to have explicit time dependence in these gauge conditions functions. The physical subspace $\Sigma = \Sigma(t)$ evolves with changing time in the superspace of 3-metrics and induces the evolution of the 3-dimensional spacelike slice $\sigma(t)$ in 4-dimensional spacetime,

$$\Sigma(t), \frac{\partial \Sigma(t)}{\partial t} \neq 0 \quad \Rightarrow \quad \sigma(t), \frac{\partial \sigma(t)}{\partial t} \neq 0, \frac{\partial e^\alpha(\mathbf{x}, t)}{\partial t} = N^\alpha(\mathbf{x}, t) \sim N^\mu = -J^{-1}{}^\mu_\nu \frac{\partial \chi^\nu}{\partial t} \neq 0. \tag{5.56}$$

This is the case of quantum cosmology [*see A.Barvinsky, Unitarity approach to quantum cosmology, Phys. Rept. 230 (1993) 237-367, DOI: 10.1016/0370-1573(93)9003*], which goes beyond this lecture course. Instead of it, consider the case of linearized GR in asymptotically-flat spacetime with $H_0 \neq 0$, where the mechanism of this evolution is quite different.

We have linearized GR on flat spacetime background with the metric and conjugated momenta perturbations

$$\gamma_{ij} = \delta_{ij} + h_{ij}, \quad \pi^{ij} = p^{ij}, \quad \varepsilon \equiv (\gamma_{ij}, p^{ij}) \ll 1, \tag{5.57}$$

$$N = 1 + n, \quad N^i = n^i, \quad (n, n^i) \ll 1, \tag{5.58}$$

and the linearized Hamiltonian and momenta constraints $H_\mu = H^{(1)}_\mu + O(\varepsilon^2)$:

$$H^{(1)}_\perp = \Delta h_{ii} - \partial^i \partial^j h_{ij}, \quad \partial^i \equiv \delta^{ij} \partial_j \tag{5.59}$$

$$H^{(1)}_i = -2\partial^j p_{ij}. \tag{5.60}$$

Problem 5.3. Derive these expressions

Problem with the *coordinate* gauge conditions $\chi^\mu(h_{ij})$: the Faddeev-Popov operator in the linearized theory is degenerate, $J^{(1)\mu}_\perp = \{\chi^\mu, H^{(1)}_\perp\} = 0$ because $H^{(1)}_\perp$ is independent of canonical momenta. Solution – replacement by *phase-space* gauge conditions depending on both momenta and coordinates of the GR phase space,

$$\chi^\mu(h_{ij}) \Rightarrow \chi^\mu(h_{ij}, p^{ij}), \tag{5.61}$$

$$\chi^\perp = \delta_{ij} p^{ij}, \quad \chi^i = \partial_j h^{ij}. \tag{5.62}$$

The Faddeev-Popov operator is now invertible

$$J^\mu_\nu = \{\chi^\mu, H^{(1)}_\nu\} = \begin{bmatrix} -2\Delta & 0 \\ 0 & \Delta\delta^i_j + \partial^i\partial_j \end{bmatrix} \delta(\mathbf{x}, \mathbf{y}), \quad \mu \mapsto (\perp, \mathbf{x}; i, \mathbf{x}), \ \nu \mapsto (\perp, \mathbf{y}; j, \mathbf{y}), \tag{5.63}$$

where the entries of the block two-by-two matrix correspond to $\perp$ and i discrete indices within the condensed indices μ.

Constraints and gauge conditions $\chi^\mu = 0$, $H^{(1)}_\nu = 0$ or $\Delta h = 0$, $\partial^j p_{ij} = 0$ imply that metric and momentum perturbations are transverse-traceless tensors (uniqueness of the solution $h = 0$ of $\Delta h = 0$ is guaranteed by zero boundary condition at spatial infinity $h\,|_\infty = 0$,

$$h_{ij} = h^{TT}_{ij}, \ \ p^{ij} = p^{ij}_{TT}. \tag{5.64}$$

Equations for perturbations of lapse and shift functions $J^\mu_\nu n^\nu = 0$ (or $\Delta n^\mu(\mathbf{x}) = 0$) also have zero solution in view of the same Dirichlet boundary conditions $n^\nu|_\infty = 0$. Thus, boundary conditions $N^\mu|_\infty = \delta^\mu_\perp$ enforce nonzero lapse function – the gravitational Lagrange multiplier – even despite time-independent gauge conditions.

The same guarantees nonzero physical Hamiltonian H_0 in asymptotically-flat spacetime. As it was discussed in previous lectures, the Gibbons-Hawking surface term in GR action, which plays the role of nonvanishing Hamiltonian H_0, should guarantee cancellation of the second-order derivatives normal to the timelike (side) boundary in the 3-curvature scalatr ${}^{(3)}R = \partial^i\partial^j h_{ij} - \Delta h + \ldots$ of the total ADM Lagrangian of the theory. Bearing in mind that in asymptotically flat spacetime $N|_{|\mathbf{x}|\to\infty} \to 1$ one has by integrating by parts

$$L_{ADM} = \int d^3\mathbf{x}\,\sqrt{\gamma}\,N({}^{(3)}R + \ldots) - H_0 = \int d^3\mathbf{x}\,(\text{no second}-\text{order derivatives})$$
$$+ \int\limits_{|\mathbf{x}|\to\infty} d^2\sigma^i(\partial^j h_{ij} - \partial_i h) - H_0. \tag{5.65}$$

This means that H_0 should coincide with the surface term here, which is equivalent to the following expression,

$$H_0 = \int\limits_{|\mathbf{x}|\to\infty} d^2\sigma^i(\partial^j h_{ij} - \partial_i h) = \int d^3\mathbf{x}\,(\partial^i\partial^j h_{ij} - \Delta h) = \int d^3\mathbf{x}\,\Big(H^{(2)}_\perp + O(\varepsilon^2)\Big). \tag{5.66}$$

In the last equality $H^{(2)}_\perp$ is the part of the the full Hamiltonian constraint quadratic in $\varepsilon = (\gamma_{ij}, p^{ij})$, and we used the fact that

$$H_\perp = H^{(1)}_\perp + H^{(2)}_\perp + O(\varepsilon^2) = -\partial^i\partial^j h_{ij} + \Delta h + H^{(2)}_\perp + O(\varepsilon^2) = 0.$$

Thus the total action of the linearized GR (quadratic in perturbations of two physical polarizations of the gravitational field) is

$$S^{(2)}[\,h_{TT}, p_{TT}\,] = \int dt \int d^3\mathbf{x}\,\big[\,p^{ij}_{TT}\dot h^{TT}_{ij} - H^{(2)}_\perp(h_{TT}, p_{TT})\,\big]. \tag{5.67}$$

Problem 5.4. Prove: $\int d^3\mathbf{x}\,H^{(2)}_\perp(h_{TT}, p_{TT}) = \int d^3\mathbf{x}\,\big[\,(p^{ij}_{TT})^2 + (\partial_k h^{TT}_{ij})^2\big]$ and derive Lagrangian equations of motion: $(-\partial_t^2 + \Delta)h^{TT}_{ij} \equiv \Box h^{TT}_{ij} = 0$

Thus the physical Hamiltonian of general relativity gets localized in the spacetime bulk even despite its origin from the surface integral at the boundary of space sections.

Lecture 6. Quantization: from Quantum Mechanics to QFT

- Canonical quantization: a reminder
- QFT: normal ordering and the functional formulation of Wick theorem
- Spacetime condensed notations
- Interacting fields and interaction picture representation

6.1 Canonical quantization: a reminder

We begin by reminding basic principles of canonical quantization in quantum mechanics. Quantization of a dynamical system with phase space variables $q = q^i,\ p = p_i$ and the Hamiltonian $H(q,p)$ consists in promoting these classical quantities to Hermitian operators

$$q, p, H(q,p) \to \hat q, \hat p, \hat H, \tag{6.1}$$

acting in the Hilbert space of states $|\,\psi\,\rangle$ with a positive norm $\langle\,\psi\,|\,\psi\,\rangle > 0$. The phase space operators satisfy canonical commutation relations

$$[\,\hat{q}^i, \hat{p}_k\,] \equiv \hat{q}^i \hat{p}_k - \hat{p}_k \hat{q}^i = i\hbar\delta^i_k, \tag{6.2}$$

which are postulated as a promotion of classical Poisson brackets to the quantum level, $\{q,p\} \to [\,\hat{q},\hat{p}\,] = i\hbar\widehat{\{q,p\}} = i\hbar 1$. Similarly, the commutator of other quantum operators $\hat{\mathcal{O}} = \widehat{\mathcal{O}(q,p)}$ in the leading order in the Planck constant $\hbar$ follow from the operator realization of their classical Poisson brackets

$$[\,\hat{\mathcal{O}}_1, \hat{\mathcal{O}}_2\,] = i\hbar\widehat{\{\mathcal{O}_1, \mathcal{O}_2\}} + O(\hbar^2), \tag{6.3}$$

where $O(\hbar^2)$ depend of course on the details of of the operator realization $O(q,p) \to \widehat{\mathcal{O}(q,p)}$ and $\{\mathcal{O}_1, \mathcal{O}_2\} \to \widehat{\{\mathcal{O}_1, \mathcal{O}_2\}}$.

Physical evolution in time is encoded in time dependent quantum state $|\,\psi(t)\,\rangle$, which satisfies the Schroedinger equation

$$i\hbar\,\frac{d}{dt}\,|\,\psi(t)\,\rangle = \hat{H}\,|\,\psi(t)\,\rangle. \tag{6.4}$$

For conserved Hamiltonians explicitly independent of time a formal solution reads in terms of unitary evolution operator $\hat{U}(t)$

$$|\,\psi(t)\,\rangle = \hat{U}(t)\,|\,\psi(0)\,\rangle, \quad \hat{U}(t) = e^{-i\hat{H}\,t/\hbar}, \quad \hat{U}^\dagger(t)\,\hat{U}(t) = \mathbb{I}. \tag{6.5}$$

This operator intertwines the original Schroedinger operators with the Heisenberg operators satisfying the Heisenberg equations of motion. I.e. for $\hat{q}$ it looks like

$$\hat{q}_H(t) = \hat{U}^\dagger(t)\,\hat{q}\,\hat{U}(t), \tag{6.6}$$

$$i\hbar\frac{d}{dt}\hat{q}_H(t) = [\,\hat{q}_H(t), \hat{H}\,], \tag{6.7}$$

so that the Heisenberg equations of motion follow from the classical ones by the generalization of the Poisson bracket to the quantum commutator with the Hamiltonian, $\{...,H\} \to [...,\hat{H}\,]/i\hbar$. **In what follows we will basically use the units with $\hbar = 1$.**

Consider now free systems with the quadratic actions and Hamiltonians of the form

$$S[\,q\,] = \frac{1}{2}\int dt\,\Big[a_{ik}\dot{q}^i\dot{q}^k - (\omega^2)_{ik}q^iq^k\Big], \tag{6.8}$$

$$H(q,p) = \frac{1}{2}\,p_i a^{ik} p_k + \frac{1}{2}\,(\omega^2)_{ik}q^iq^k. \tag{6.9}$$

In the simplest case a_{ik} is just a unit matrix (canonical normalization), and the matrix of the quadratic potential is diagonal in frequencies of normal modes

$$a_{ik} = \delta_{ik}, \quad (\omega^2)_{ik} = \mathrm{diag}\,(\omega_i^2), \quad \omega_i^2 > 0. \tag{6.10}$$

Then normal modes satisfy harmonic oscillator equation, $\ddot{q}^i + \omega_i^2 q^i = 0$. It has a solution which we write directly for Heisenberg operators

$$\hat{q}^i_H(t) = \sum_A \Big(u^i_A(t)\,\hat{a}_A + u^{i*}_A(t)\,\hat{a}^\dagger_A\Big), \quad u^i_A(t) = \frac{1}{\sqrt{2\omega_i}}e^{-i\omega_i t}\delta^i_A, \quad u^{i*}_A(t) = \frac{1}{\sqrt{2\omega_i}}e^{i\omega_i t}\delta^i_A. \tag{6.11}$$

It is convenient not to identify the index A enumerating the basis functions with the label m of the phase space coordinate q^i (this will be clear later) and consider $u(t) \equiv u^i_A(t)$ as a matrix. This is a decomposition in positive and negative frequency basis functions – two complex conjugated sets $u^i_A(t)$ and $u^{i*}_A(t)$ which span the space of all solutions. For any two solutions of the equation $\ddot{u}^i + \omega_i^2 u^i = 0$ one can introduce the conserved (but not positive definite) inner product,

$$\langle\,u_1, u_2\,\rangle \equiv i(u_1^\dagger \dot{u}_2 - \dot{u}_1^\dagger u_2) \equiv i\Big(u_1^{i*}\delta_{ik}\,\dot{u}_2^k - \dot{u}_1^{i*}\delta_{ik}\,u_2^k\Big), \quad \frac{d}{dt}\langle\,u_1, u_2\,\rangle = 0, \tag{6.12}$$

which obviously satisfies the relation $\langle u_1^*, u_2^* \rangle = -\langle u_2, u_1 \rangle$. This inner product is conserved in virtue of equations of motion and is used to split the whole space of solutions into those with the positive and negative "norms",

$$\langle u_A, u_B \rangle \equiv i\Big(u_A^{i*}\delta_{ik}\,\dot u_B^k - \dot u_A^{i*}\delta_{ik}\,u_B^k\Big) = \delta_{AB}, \quad \langle u_A^*, u_B^* \rangle = -\delta_{AB}, \quad \langle u_A^*, u_B \rangle = 0, \tag{6.13}$$

Their coefficients in the decomposition of $\hat q_H(t)$ – the two sets of Hermitian conjugated annihilation and creation operators $\hat a_A$ and $\hat a_A^\dagger$ – satisfy in virtue of the canonical commutators for $\hat q$ and $\hat p$ the commutation relations

$$[\,\hat a_A, \hat a_B^\dagger\,] = \delta_{AB} \tag{6.14}$$

Problem 6.1. Prove this commutation relation and derive:

$$\hat a = \langle u, \hat q_H \rangle = \frac{\omega \hat q + i\hat p}{\sqrt{2\omega}}, \quad \hat a^\dagger = \langle u^*, \hat q_H \rangle = \frac{\omega \hat q - i\hat p}{\sqrt{2\omega}},$$

where $\hat q = \hat q_H(0)$ and $\hat p = \hat p_H(0)$ – Schroedinger operators of coordinates and momenta (which can be treated as initial conditions for Heisenberg operators) and we skip the indices i and A which are identified in this particular case with $u_A^i \sim \delta_A^i$.

Coordinate representation in the Hilbert space of $\Psi(q) = \langle\, q \,|\, \Psi \,\rangle$,

$$\hat q\,|\,q\,\rangle = q\,|\,q\,\rangle, \quad \hat p\,\Psi(q) = \frac{\hbar}{i}\frac{\partial}{\partial q}\Psi(q), \quad \langle\,\Psi\,|\,\Phi\,\rangle = \int dq\,\Psi^*(q)\,\Phi(q) \tag{6.15}$$

in the case of $\omega \neq 0$ can be replaced by the occupation number representation consisting of the vacuum state $|\,0\,\rangle$ and the tower of Fock states $|\,N\,\rangle \equiv |\,N_1, N_2, ...\,\rangle$

$$\hat a_A\,|\,0\,\rangle = 0, \quad |\,N\,\rangle = \prod_A \frac{(\hat a_A^\dagger)^{N_A}}{\sqrt{N_A!}}\,|\,0\,\rangle, \quad N = N_1, N_2, ..., \tag{6.16}$$

$$\langle\, q\,|\,0\,\rangle = \Psi_0(q) = \prod_i \Big(\frac{\omega_i}{\pi}\Big)^{1/4} e^{-\omega_i (q^i)^2/2} = \Big(\prod_i \frac{\omega_i}{\pi}\Big)^{1/4} \exp\Big[-\frac{1}{2} q^i \omega_{ik} q^k\Big]. \tag{6.17}$$

In this representation the Hamiltonian is diagonal

$$\hat H = \frac{1}{2}\sum_i \big(\hat p_i^2 + \omega_i^2 (\hat q^i)^2\big) = \sum_A \Big(\omega_A\,\hat a_A^\dagger \hat a_A + \frac{1}{2}\Big), \tag{6.18}$$

$$\langle\, M\,|\,\hat H\,|\,N\,\rangle = \delta_{MN}\sum_A \Big(\omega_A N_A + \frac{1}{2}\Big), \quad \delta_{MN} = \prod_A \delta_{M_A, N_A}. \tag{6.19}$$

Note: for the modes with $\omega_A = 0$ occupation number representation associated with the Hamiltonian diagonalization does not exist, and therefore another representation is needed – usually this is a finite set of ***zero*** modes, for which the coordinate representation is used.

6.2 QFT: normal ordering and the functional formulation of Wick theorem

Formal transition from quantum mechanics to QFT with infinite number of degrees of freedom consists in extending the range of the index of q^i to include both discrete spin-tensor labels and spatial coordinates. Say, for a scalar field $\varphi(\mathbf{x})$ and its canonical momentum $\pi(\mathbf{x})$ this means

$$q^i = \varphi(\mathbf{x}), \quad p_i = \pi(\mathbf{x}), \quad i \mapsto \mathbf{x}. \tag{6.20}$$

The index i becomes condensed and, according to the DeWitt rule, contraction of such repeated condensed indices implies integration over their continuous part, i. e. over spatial coordinates, the derivatives with respect to phase space variables become 3-dimensional functional derivatives and the Poisson brackets respectively read

$$\sum_i \mapsto \int d^3\mathbf{x}, \quad \frac{\partial}{\partial q^i} = \frac{\delta}{\delta\varphi(\mathbf{x})}, \quad \{\mathcal{O}_1, \mathcal{O}_2\} = \int d^3\mathbf{x}\,\Big(\frac{\delta\mathcal{O}_1}{\delta\varphi(\mathbf{x})}\frac{\delta\mathcal{O}_2}{\delta\pi(\mathbf{x})} - (1 \Leftrightarrow 2)\Big). \tag{6.21}$$

Correspondingly the massive scalar field action

$$S[\,\varphi\,] = \frac{1}{2}\int d^4x\,\big(-\partial_\mu\varphi\partial^\mu\varphi - m^2\varphi^2\big) = \frac{1}{2}\int dt\int d^3\mathbf{x}\,\big[\,\dot{\varphi}^2(\mathbf{x}) - \varphi(\mathbf{x})(m^2-\Delta)\varphi(\mathbf{x})\,\big] \tag{6.22}$$

takes the form (6.8) with the functional matrices

$$a_{ik} = \delta(\mathbf{x},\mathbf{y}),\quad \omega_{ik} = \sqrt{m^2-\Delta}\,\delta(\mathbf{x},\mathbf{y}),\quad i\mapsto\mathbf{x},\quad k\mapsto\mathbf{y}. \tag{6.23}$$

Canonical commutation relations (6.2) with $\delta^i_k = \delta(\mathbf{x},\mathbf{y})$ read $[\,\hat{\varphi}(\mathbf{x}),\hat{\pi}(\mathbf{y})\,] = i\delta(\mathbf{x},\mathbf{y})$, and the decomposition of the Heisenberg operators into positive-negative frequency parts looks as (6.11) with the basis functions – solutions of the Klein-Gordon equation,

$$u^i_A(t) = \frac{1}{(2\pi)^{3/2}\sqrt{2\omega_\mathbf{p}}}e^{-i\omega_\mathbf{p}t - i\mathbf{p}\mathbf{x}},\quad \omega_\mathbf{p} = \sqrt{\mathbf{p}^2+m^2},\quad i\mapsto\mathbf{x},\quad A\mapsto\mathbf{p}. \tag{6.24}$$

Note that A is also a condensed index, its contraction in (6.11) implies integration over $\mathbf{p}$, $\sum_A = \int d^3\mathbf{p}$, and $u^i_A(t)$ is a functional matrix with respect to its two continuous entries $i=\mathbf{x}$ and $A=\mathbf{p}$. Commutation relations for creation-annihilation operators $[\,\hat{a}_\mathbf{p},\hat{a}^\dagger_{\mathbf{p}'}\,] = \delta(\mathbf{p},\mathbf{p}')$.

In the occupation number representation any operator $\hat{\mathcal{O}}$ which can be expanded in powers of the $\hat{q}$ and $\hat{p}$ can be rewritten as power series in $\hat{a}$ and $\hat{a}^\dagger$. Then by their commutation it can be converted to the **normal form** in which all creation operators stand to the left of annihilation ones

$$\hat{\mathcal{O}} =\,:\mathcal{O}(\hat{a}^*,\hat{a}):\; = \sum_{m,n}\mathcal{O}^{A_1...A_m,B_1...B_n}_{m,n}\,\hat{a}^\dagger_{A_1}...\hat{a}^\dagger_{A_m}\,\hat{a}_{B_1}...\hat{a}_{B_n}, \tag{6.25}$$

where $\mathcal{O}^{A_1...A_m,B_1...B_n}_{m,n}$ are some coefficients of the operator monomials. The c-number function,

$$\mathcal{O}(a^*,a) = \sum_{m,n}\mathcal{O}^{A_1...A_m,B_1...B_n}_{m,n}\,a^*_{A_1}...a^*_{A_m}\,a_{B_1}...a_{B_n}, \tag{6.26}$$

is then called the **normal symbol** of $\hat{\mathcal{O}}$. The action of such operator on the vacuum state and its expectation value in the vacuum state obviously express like

$$:\mathcal{O}(\hat{a}^*,\hat{a}):|\,0\,\rangle = \sum_{m=0}^\infty\mathcal{O}^{A_1...A_m}_{m,0}\,\hat{a}^\dagger_{A_1}...\hat{a}^\dagger_{A_m}\,|\,0\,\rangle, \tag{6.27}$$

$$\langle\,0\,|:\mathcal{O}(\hat{a}^*,\hat{a}):|\,0\,\rangle = \mathcal{O}_{0,0}. \tag{6.28}$$

The matrix element of the operator between two Fock states $|\,\varPsi\,\rangle = (\hat{a}^\dagger)^k|\,0\,\rangle$ and $|\,\varPhi\,\rangle = (\hat{a}^\dagger)^l|\,0\,\rangle$ equals the sum of terms resulting from the commutation of all $\hat{a}$ and $\hat{a}^\dagger$ respectively to the right and to the left. The answer reads a sum of all possible contractions of creation-annihilation pairs $[\,\hat{a}_A,\hat{a}^\dagger_B] = \delta_{AB}$ symbolically shown as

$$\langle\,\varPsi\,|:\mathcal{O}(\hat{a}^*,\hat{a}):|\,\varPhi\,\rangle = \sum_{\rm contractions}\langle\,0\,|\,\hat{a}^k\sum_{m,n}\mathcal{O}_{m,n}(\hat{a}^*)^m\hat{a}^n\,(\hat{a}^\dagger)^l|\,0\,\rangle. \tag{6.29}$$

The same applies to any monomial function of operators $\hat{\varphi}$ **linear** in creation-annihilation operators – any such monomial equals the sum of normally ordered monomials with all possible contractions between $\hat{\varphi}$'s. This is the content of the so-called Wick theorem

$$\hat{\varphi}_1...\hat{\varphi}_n =\,:\hat{\varphi}_1...\hat{\varphi}_n:\,+\sum_{\substack{\rm single\\ \rm contractions}}:\overbracket{\hat{\varphi}_1\hat{\varphi}_2}...\hat{\varphi}_n:\,+\sum_{\substack{\rm double\\ \rm contractions}}:\hat{\varphi}_1\hat{\varphi}_2\hat{\varphi}_3\hat{\varphi}_4...\hat{\varphi}_n:\,+\,...\,, \tag{6.30}$$

$$\overbracket{\hat{\varphi}_1\hat{\varphi}_2} = \hat{\varphi}_1\hat{\varphi}_2 -\!:\hat{\varphi}_1\hat{\varphi}_2:\; = \langle\,0\,|\,\hat{\varphi}_1\hat{\varphi}_2\,|\,0\,\rangle. \tag{6.31}$$

Explicitly this can be written down in the following form

$$\hat{\varphi}_1(x_1)...\hat{\varphi}_n(x_n) = \,:\, \prod_{i<k}\left(1 + \int dx\,dy\,\frac{\delta}{\delta\varphi_i(x)}D(x,y)\frac{\delta}{\delta\varphi_k(y)}\right)\,\varphi_1(x_1)...\varphi_n(x_n)\,\Big|_{\varphi\to\hat{\varphi}}\,:\,, \tag{6.32}$$

$$D(x,y) = \overbrace{\hat{\varphi}(x)\hat{\varphi}}(y) = \langle\, 0\,|\,\hat{\varphi}(x)\hat{\varphi}(y)\,|\,0\,\rangle, \tag{6.33}$$

where $D(x,y)$ is the two-point kernel of this contraction. Here the second order differential operator in functional derivatives performs all the needed contractions in the chain of operators $\hat{\varphi}_1(x_1)...\hat{\varphi}_n(x_n)$. The sequence of operations is as follows: first all operators are replaced by c-number functions, secondly their pairwise contractions are inserted, thirdly the remaining functions are again replaced by the operators which are normally ordered.

This Wick theorem can be reformulated in a more elegant (and useful, as we will later see) form for the special case of the symmetrized product of operators on the left hand side. **Functional formulation of the Wick theorem** states that any **symmetrized** monomial operator can be represented in the normally ordered form by acting on this monomial with a special exponentiated differential operator

$$\mathrm{Sym}\Big(\hat{\varphi}(x_1)...\hat{\varphi}(x_n)\Big) = \,:\, \exp\left(\frac{1}{2}\int dx\,dy\,D(x,y)\,\frac{\delta}{\delta\varphi(x)}\frac{\delta}{\delta\varphi(y)}\right)\,\varphi(x_1)...\varphi(x_n)\,\Big|_{\varphi\to\hat{\varphi}}\,:\,, \tag{6.34}$$

Here symmetrization is given by $n!$ terms with all possible permutations $(1,...n)\to\sigma(1,...n)$ of the original n entries, divided by overall $n!$ factor

$$\mathrm{Sym}f(1,...n) \equiv \frac{1}{n!}\sum_\sigma f(\sigma(1,...n)). \tag{6.35}$$

Note that this symmetrization on the left-hand side is very important, and although contraction $D(x,y)$ is generally not symmetric, only its symmetric part contributes to the right-hand side of (6.34). Rigorous proof of the theorem is somewhat lengthy and can be found in [A.N.Vasiliev, *Functional methods in quantum field theory and quantum statistics*, St Petersburg University Press, St Petersburg, 1976; Overseas Publishers Association, Amsterdam B.V., 1998]. It begins with the proof by induction of the Eq.(6.32) and then this equation takes the form of (6.34) when applied to the symmetrized monomial.

6.3 Spacetime condensed notations

Remarkably this theorem can be further reformulated in even more concise form by using **spacetime condensed notations**. In these notations we include not only spatial coordinates into the condensed index, but also absorb into this index the time itself. In contrast to the canonical formalism, this will allow us in what follows to make the formalism Lorentz covariant. As a rule we will pick up such indices from the first part of Latin alphabet. Thus, the generalization of the above technique to the fields of nonzero spin with some spin-tensor and isotopic structure, labelled by discrete indices I, looks like

$$\varphi(x) \;\Rightarrow\; \phi^I(x) = \phi^a, \quad a\mapsto(I,x), \quad x\equiv x^\mu = (x^0,\mathbf{x}). \tag{6.36}$$

DeWitt summation-integration rule in these spacetime condensed notations implies both space and time integration in contraction of their indices

$$\psi_a\phi^a = \int d^4x\,\sum_I \psi_I(x)\phi^I(x), \tag{6.37}$$

so that Wick theorem for symmetrized products of free Heisenberg operators takes simple readable form which is easy to memorize

$$\mathrm{Sym}\Big(\hat{\phi}_1...\hat{\phi}_n\Big) = \,:\, \exp\left(\frac{1}{2}D^{ab}\frac{\delta}{\delta\phi^a}\frac{\delta}{\delta\phi^b}\right)\,\phi_1...\phi_n\,\Big|_{\phi\to\hat{\phi}}\,:\,, \tag{6.38}$$

$$D^{ab} = \overbrace{\hat{\phi}^a\hat{\phi}}{}^b = \langle\, 0\,|\,\hat{\phi}^a\hat{\phi}^b\,|\,0\,\rangle \tag{6.39}$$

As mentioned above D^{ab} can be replaced here by its symmetrized version $D^{(ab)}$,

$$D^{(ab)} = \mathrm{Sym}\big(\hat\phi^a\hat\phi^b\big) - :\hat\phi^a\hat\phi^b: \tag{6.40}$$

so that when used in (6.38) this difference between two types of operator orderings (symmetrized and normal) matches with the difference of these orderings in the left and right hand sides of Eq.(6.38). The exponentiated differential operator factor serves as a transition between these two orderings. In what follows we will see how this mnemonic rule extends to other types of operator orderings (chronological and others).

In terms of spacetime condensed notations the positive-negative frequency decomposition takes the form where the matrices of basis functions serve as coefficients of transition from the "vectors" of creation-annihilation operators to Heisenberg operators $\hat\phi^a$. This in fact suggests to rewrite this relation in "supercondensed" form with all indices omitted – u is a functional matrix u^a_A, $\hat a = \hat a_A$ is a vector whose components are labelled by the index A (unfortunately it is in the subscript position, but we will leave it "covariant"),

$$\hat\phi^a = \sum_A \big(u^a_A \hat a_A + u^{a*}_A \hat a^\dagger_A\big) = u\,\hat a + u^*\hat a^\dagger. \tag{6.41}$$

Thus we have for contraction matrix

$$D^{ab} = \sum_{A,B} u^a_A\,\langle\, 0\,|\,\hat a_A \hat a^\dagger_B\,|\,0\,\rangle\, u^{b*}_B = \sum_A u^a_A\, u^{b*}_A \;\Rightarrow\; D = u\,u^\dagger, \quad u^\dagger \equiv u^{*T}, \tag{6.42}$$

where Hermitian conjugation of $u \to u^\dagger$ is understood as its complex conjugation and transposition. Note, however, that the condensed indices a and A of u^a_A are very different in nature – a is spacetime condensed because it contains time $t = x^0$, whereas A is essentially space (canonical) condensed, for this entry contains only spatial coordinates or dual spatial momentum (see above expressions for u of the scalar field),

$$u^a_A = u^i_A(t) = u_{\mathbf p}(x), \quad a \mapsto (i,t) \mapsto (\mathbf x, x^0) = x^\mu, \quad A \mapsto \mathbf p.$$

Correspondingly, functional contraction of $A = \mathbf p$ does not involve time integration – it incorporates only spatial quantum numbers summation and integration, $\sum_A \mapsto \int d^3\mathbf p$. In particular, for a scalar field the contraction function D^{ab} is the *positive-frequency Wightman* function,

$$D(x,y) = \int d^3\mathbf p\, u_{\mathbf p}(x)\, u^*_{\mathbf p}(y) = \frac{1}{(2\pi)^3}\int d^3\mathbf p\, \frac{e^{-ip(x-y)}}{2\,\omega_{\mathbf p}}\Big|_{p_0=+\omega_{\mathbf p}} \equiv i\,G^{(+)}(x,y). \tag{6.43}$$

Problem 6.2. Derive this expression and show that it gives a positive-frequency part of the Pauli-Jordan commutator function:

$$[\,\hat\varphi(x), \hat\varphi(y)\,] = i\,\tilde G(x,y), \quad \tilde G(x,y) = G^{(+)}(x,y) + G^{(-)}(x,y), \quad G^{(-)}(x,y) = [\,G^{(+)}(x,y)\,]^*, \tag{6.44}$$

6.4 Interacting fields and interaction picture representation

Thus far, it was the quantum theory of free fields. Now we turn to interacting fields with nonlinear interaction in their action functional. The first thing to do, in order to build the perturbation theory, is to split the field into the background configuration ϕ_0 and perturbation h and expand the action in powers of h, $\phi = \phi_0 + h$,

$$S[\,\phi\,] = S[\,\phi_0\,] + \frac{\delta S}{\delta\phi}\bigg|_{\phi_0} h + \frac12\,\frac{\delta^2 S}{\delta\phi\delta\phi}\bigg|_{\phi_0} h\,h + S_I[\,\phi_0, h\,], \quad S_I[\,\phi_0,h\,] = O[\,h^3\,], \tag{6.45}$$

where we use adopted above spacetime supercondensed and condensed notations, implying in particular that the second order term reads as

$$\frac12\,\frac{\delta^2 S}{\delta\phi\delta\phi}\bigg|_{\phi_0} h\,h \equiv \frac12\,\frac{\delta^2 S}{\delta\phi^a\delta\phi^b}\bigg|_{\phi_0} h^a\,h^b \equiv \frac12\int d^4x\,d^4y\;\frac{\delta^2 S}{\delta\phi^I(x)\delta\phi^J(y)}\bigg|_{\phi_0} h^I(x)\,h^J(y). \tag{6.46}$$

If one chooses ϕ_0 as some fixed solution of classical equations of motion $\delta S/\delta\phi|_{\phi_0} = 0$ and treats $S[\,\phi_0\,]$ as an inessential constant, then the action of a new field h begins with the quadratic order and contains a nonlinear interaction term $O(\,h^3)$. The transition to the canonical formalism of this field, $h \to (q,p)$ correspondingly results in the total Hamiltonian consisting of a quadratic part H_0 and the interaction part V beginning with the term cubic in (q,p). Therefore, at the quantum level this Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{V}, \tag{6.47}$$

drives the Schroedinger evolution of the quantum state

$$i\hbar\,\frac{d}{dt}\,|\,\psi(t)\,\rangle = (\hat{H}_0 + \hat{V})\,|\,\psi(t)\,\rangle. \tag{6.48}$$

The ansatz for the solution of this equation in the form

$$|\,\psi(t)\,\rangle = e^{-i\hat{H}_0 t}\,|\,\psi_I(t)\,\rangle, \tag{6.49}$$

allows one to go over to the interaction picture representation in which the quantum state $|\,\psi_I(t)\,\rangle$ satisfies the Schroedinger equation

$$i\hbar\,\frac{d}{dt}\,|\,\psi_I(t)\,\rangle = \hat{V}_I(t)\,|\,\psi_I(t)\,\rangle \tag{6.50}$$

with the interaction Hamiltonian $\hat{V}_I(t)$ in this representation

$$\hat{V}_I(t) = e^{i\hat{H}_0 t}\,\hat{V}(\hat{q},\hat{p})\,e^{-i\hat{H}_0 t} = \hat{V}(\hat{q}_I(t),\hat{p}_I(t)), \tag{6.51}$$

$$\hat{q}_I(t) = e^{i\hat{H}_0 t}\,\hat{q}\,e^{-i\hat{H}_0 t}, \quad \hat{p}_I(t) = e^{i\hat{H}_0 t}\,\hat{p}\,e^{-i\hat{H}_0 t}. \tag{6.52}$$

The *time-dependent* phase space operators $\hat{q}_I(t)$ and $\hat{p}_I(t)$ of the interaction picture representation satisfy the equations of the linearized theory

$$i\frac{d}{dt}\hat{q}_I(t) = [\,\hat{q}_I(t),\hat{H}_0\,], \quad i\frac{d}{dt}\hat{p}_I(t) = [\,\hat{p}_I(t),\hat{H}_0\,]. \tag{6.53}$$

Since $[\,\hat{V}_I(t),\hat{V}_I(t')\,] \neq 0$, the solution for $|\,\psi_I(t)\,\rangle$ is more complicated than a simple exponentiation of $\int dt\,\hat{V}(t)$. It is the evolution operator $\hat{U}(t,t_-)$ in the interaction picture representation,

$$|\,\psi_I(t)\,\rangle = \hat{U}(t,t_-)\,|\,\psi_I(t_-)\,\rangle, \tag{6.54}$$

$$\hat{U}(t,t_-) = e^{i\hat{H}_0 t}\,e^{-i\hat{H}(t-t_-)}\,e^{-i\hat{H}_0 t_-}, \tag{6.55}$$

which is given by the chronologically ordered T-exponent,

$$\begin{aligned}\hat{U}(t,t_-) &= \mathbb{T}\,\exp\Big(-i\int_{t_-}^{t} dt'\,\hat{V}_I(t')\Big)\\ &= \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{t_-}^{t}dt_1\int_{t_-}^{t}dt_2...\int_{t_-}^{t}dt_n\,\mathbb{T}\Big(\hat{V}_I(t_1)\hat{V}_I(t_2)...\hat{V}_I(t_n)\Big).\end{aligned} \tag{6.56}$$

Chronological ordering symbol $\mathbb{T}$ means that all factors are ordered from right to left in the direction of growing value of their time argument, operators at later times standing to the left of those at earlier times. Formally this can be written down as a sum over all possible $n!$ permutations of positions of factors – each term weighted by 1 or 0, depending on the order of time arguments

$$\mathbb{T}\Big(\hat{V}_I(t_1)\hat{V}_I(t_2)...\hat{V}_I(t_n)\Big) = \sum_{\sigma(1,2,...n)}\theta(t_1-t_2)\theta(t_2-t_3)...\theta(t_{n-1}-t_n)\hat{V}_I(t_1)\hat{V}_I(t_2)...\hat{V}_I(t_n), \tag{6.57}$$

where $\theta(x)$ is a step function

$$\theta(x) = \left\{\begin{array}{ll}1, & x>0,\\ 0, & x<0.\end{array}\right. \tag{6.58}$$

For $n=2$, for example, this product reads

$$\mathbb{T}\Big(\hat{V}_I(t_1)\hat{V}_I(t_2)\Big) = \theta(t_1-t_2)\,\hat{V}_I(t_1)\hat{V}_I(t_2) + \theta(t_2-t_1)\,\hat{V}_I(t_2)\hat{V}_I(t_1). \tag{6.59}$$

For coincident time arguments the ordering prescription should be additionally fixed, which basically implies the specification of the value of $\theta(x)$ at $x=0$. This, however, will not be important for us in what follows, because this is the set of points of measure zero, which is responsible for ultraviolet divergences and renormalization to be separately considered.

Properties of the T-exponent:

1) $\hat{U}(t,t_-)$ satisfies the composition law $\hat{U}(t,t_-)=\hat{U}(t,t_1)\hat{U}(t_1,t_-)$. This can be proven by calculating $\partial_{t_1}\big(\hat{U}(t,t_1)\hat{U}(t_1,t_-)\big)=0$.

2) Unitarity, $\hat{U}^\dagger(t,t_-)\hat{U}(t,t_-)=\hat{1}$ – follows from Hermiticity of $\hat{V}_I(t)$.

3) Variational law under $\hat{V}_I(t)\to\hat{V}_I(t)+\delta\hat{V}_I(t)$,

$$\delta\hat{U}(t,t_-) = -i\int_{t_-}^{t} dt_1\,\hat{U}(t,t_1)\,\delta\hat{V}_I(t_1)\,\hat{U}(t_1,t_-) \tag{6.60}$$

This can be proven by solving the next problem.

Problem 6.3. Prove that (6.56) solves the Cauchy problem for the unitary evolution operator $\hat{U}(t,t_-)$

$$i\frac{d}{dt}\hat{U}(t,t_-) = \hat{V}_I(t)\,\hat{U}(t,t_-),\quad \hat{U}(t_-,t_-)=\hat{1} \tag{6.61}$$

and the above variational equation.

Hint 1: use complete symmetry of $\mathbb{T}\big(\hat{V}_I(t_1)\hat{V}_I(t_2)...\hat{V}_I(t_n)\big)$ with respect to permutations of $t_1,...t_n$ and the fact that the integration measure in (6.56) can be replaced by $\int_{t_-}^{t} dt_1...dt_n = n!\int_{t_-}^{t}dt_1\int_{t_-}^{t_1}dt_2...\int_{t_-}^{t_{n-1}}dt_n$.

Hint 2: Prove by integrating Eq.(6.61) that $\hat{U}(t,t_-)=\hat{1}-i\int_{t_-}^{t}dt'\,\hat{V}_I(t'),\hat{U}(t',t_-)$ and solve this integral equation by iterations – that is by systematically substituting the solution from the previous perturbation step into the right hand side of this equation,

$$\hat{U}(t,t_-) = \hat{1} + \sum_{n=1}^{\infty}(-i)^n\int_{t_-}^{t}dt_1\int_{t_-}^{t_1}dt_2...\int_{t_-}^{t_{n-1}}dt_n\,\mathbb{T}\Big(\hat{V}_I(t_1)\hat{V}_I(t_2)...\hat{V}_I(t_n)\Big). \tag{6.62}$$

Let us now choose a special type of perturbation which is linear in the operators of phase space variables, $\delta\hat{V}_I(t)=-\delta j_a(t)\hat{\phi}_I^a(t)\equiv-\delta J_i(t)\hat{q}_I^i-\delta I^i(t)\hat{p}_i^I(t)$, with the c-number coefficients – the variation of the sources

$$j_a(t) = J_i(t), I^i(t), \tag{6.63}$$

dual to the full set of interaction picture operators $\hat{\phi}^a(t)=\hat{q}_I^i(t),\hat{p}_i^I(t)$. Then

$$\frac{1}{i}\frac{\delta\hat{U}(t_+,t_-)}{\delta j(t_1)} = \hat{U}(t_+,t_1)\,\hat{\phi}_I(t_1)\,\hat{U}(t_1,t_-) = \mathbb{T}\Big(\hat{\phi}_I(t_1)\,\hat{U}(t_+,t_-)\Big), \tag{6.64}$$

and similarly the n-th order functional derivative gives

$$\begin{aligned}\frac{1}{i^n}\frac{\delta^n\hat{U}(t_+,t_-)}{\delta j(t_1)...\delta j(t_n)} &= \theta(t_1-t_2)\theta(t_2-t_3)...\theta(t_{n-1}-t_n)\,\hat{U}(t_+,t_1)\,\hat{\phi}_I(t_1)\,\hat{U}(t_1,t_2)\hat{\phi}_I(t_2)...\hat{\phi}_I(t_n)\,\hat{U}(t_n,t_-)\\ &+\text{permutations}\,(1,2,...n) = \mathbb{T}\Big(\hat{\phi}_I(t_1)...\hat{\phi}_I(t_n)\,\hat{U}(t_+,t_-)\Big).\end{aligned} \tag{6.65}$$

If we use the *spacetime condensed* index $a=(i,t)$ which includes the time label, then this relation takes a very concise form

$$\frac{1}{i^n}\frac{\delta^n\hat{U}(t_+,t_-)}{\delta j_{a_1}...\delta j_{a_n}} = \mathbb{T}\big(\hat{\phi}_I^{a_1}...\hat{\phi}_I^{a_n}\,\hat{U}(t_+,t_-)\big). \tag{6.66}$$

Note that the chronological product of operators is by definition completely symmetric with respect to permutations of their condensed indices a – time arguments and associated spatial and spin labels, which is consistent with commutativity of variational derivatives here.

Lecture 7. S-matrix and canonical Schwinger-Dyson equation

- S-matrix
- Schwinger-Dyson equations in the canonical form
- Solution of Schwinger-Dyson equations and the canonical path integral
- Calculation of the path integral via the Gaussian functional integral

Now we are ready to formulate the notion of S-matrix and set up the scattering problem. We assume that interaction switches off in distant past and distant future labeled by $t_\pm \to \pm\infty$. Therefore, in these asymptotic limits the theory is described by its linearized approximation with the free Hamiltonian $\hat{H}_0$. Its physical states in the interaction picture representation are free multi-particle states of the Fock space (6.16)

$$|\,A_1,...A_n\,\rangle = \hat{a}^\dagger_{A_1}...\hat{a}^\dagger_{A_n}|\,0\,\rangle. \tag{7.1}$$

As $|\,\psi_I(t)\,\rangle$ evolves in time by the evolution operator (6.54), the transition from the remote past state to the final future state is governed by the S-matrix

$$|\,\psi_I(+\infty)\,\rangle = \hat{S}\,|\,\psi_I(-\infty)\,\rangle, \tag{7.2}$$

$$\hat{S} = \lim_{t_\pm\to\infty}\hat{U}(t_+,t_-). \tag{7.3}$$

The amplitudes of transition from the initial multi-particle state to the final one are then given by the matrix elements

$$\langle\,B_1,...B_m\,|\,\hat{S}\,|\,A_1,...A_n\,\rangle, \tag{7.4}$$

which define the probabilities of transitions $P_{\{A\}\to\{B\}} = |\langle\,B_1,...B_m\,|\,\hat{S}\,|\,A_1,...A_n\,\rangle|^2$ and relevant measurable cross-sections of particle scattering.

Note that according to (6.55) the S-matrix factorizes into the operator product of two "half" S-matrices

$$\hat{S} = \Omega^\dagger(+\infty)\,\Omega(-\infty), \tag{7.5}$$

$$\hat{\Omega}(t) = e^{i\hat{H}t}\,e^{-i\hat{H}_0 t}. \tag{7.6}$$

The unitary operator $\Omega^\dagger(t) = \Omega^{-1}(t)$ intertwines the Heisenberg and interaction picture representations

$$\hat{\phi}_H(t) = \hat{\Omega}(t)\hat{\phi}_I(t)\hat{\Omega}^{-1}(t). \tag{7.7}$$

The existence of well-defined limits of this operator at $t \to \pm\infty$ can be intuitively explained by the fact of adiabatic switching the interaction off or turning it on in these asymptotic limits, $\hat{H} \to \hat{H}_0$, so that a naive infinity of the phase of $\hat{\Omega}(t)$ in (7.6) cancels out. The operators $\Omega(\mp\infty)$ are responsible for turning on/switching off the interaction during the half-evolution from $t = -\infty$ to $t = 0$ or half-evolution from $t = 0$ to $t = +\infty$. The moment $t = 0$ is the point at which all three types of operators – Schroedinger, interaction and Heisenberg – coincide, $\hat{\phi} = \hat{\phi}_I(0) = \hat{\phi}_H(0)$, and the interaction is fully enforced. These operators also relate the asymptotic states $|\,\{A\}\,\rangle = |\,A_1,...A_n\,\rangle$ to the so-called *'in'/'out' states* $|\,\{A\},\pm\,\rangle$,

$$|\,\{A\},\pm\,\rangle = \hat{\Omega}(\mp\infty)\,|\,\{A\}\,\rangle, \tag{7.8}$$

which are exact eigenstates of the full Schroedinger picture Hamiltonian $\hat{H}$, $(\hat{H} - E_{\{A\}})|\,\{A\},\pm\,\rangle = 0$. These two sets of solutions of the stationary Schroedinger equation are distinguished by the fact that, when evolved by the full evolution operator respectively to $t \to \mp\infty$, they tend to free quantum states with the same quantum numbers $\{A\}$ including their relevant energy $E_{\{A\}}$, $e^{-i\hat{H}t}|\,\{A\},\pm\,\rangle \to e^{-iE_{\{A\}}t}|\,\{A\}\,\rangle$. All this holds when the spectrum of the full Hamiltonian $\hat{H}$ coincides with that of the free one $\hat{H}_0$, which is a typical assumption of perturbation theory in $\hat{V}$. The details of the 'in'/'out' states and their role in the relation between the so-called *old-fashioned* or *stationary* scattering perturbation theory and the modern *time-dependent* perturbation theory of S-matrix can be found in [S.Weinberg, *The Quantum Theory of Fields*, Cambridge University Press, Cambridge, 1993, volume 1, Section 3]. Here we proceed with the time-dependent perturbation theory which is capable of maintaining the manifest spacetime covariance.

7.1 Schwinger-Dyson equations in the canonical form

Here we introduce Schwinger-Dyson equations for the generating functional of chronological products of Heisenberg operators. Schwinger-Dyson equations are a consequence of equations of motion for these operators. The solution of Schwinger-Dyson equations allows one to construct the path integral representation for the S-matrix, and this is the goal of our further work.

To begin with note that in view of (7.7) and the fact that $\hat U(t,t')=\hat\Omega^\dagger(t)\hat\Omega(t')$

$$\mathbb{T}\big(\hat\phi_I(t_1)...\hat\phi_I(t_n)\hat S\big)=\hat\Omega^\dagger(\infty)\,\mathbb{T}\big(\hat\phi_H(t_1)...\hat\phi_H(t_n)\big)\,\hat\Omega(-\infty). \tag{7.9}$$

This relation obviously applies to any operator functional $F[\,\hat\phi_I\,]$ expandable in powers of its argument and chronologically ordered. For a particular choice of $F[\,\phi\,]=\exp\big(\,i\int dt\,j(t)\phi(t)\big)\equiv\exp\big(i\,j_a\phi^a\big)$ it reads

$$\mathbb{T}\Big\{\exp\Big(\,i\int_{-\infty}^{+\infty}dt\,j(t)\hat\phi_I(t)\Big)\,\hat S\,\Big\}=\hat\Omega^\dagger(\infty)\,\mathbb{T}\exp\Big(\,i\int_{-\infty}^{+\infty}dt\,j(t)\hat\phi_H(t)\Big)\,\hat\Omega(-\infty). \tag{7.10}$$

What stands on the right hand side is the limit $t_\pm\to\pm\infty$ of the same chronologically ordered functional of *Heisenberg* operators

$$\hat Z(t_+,t_-)=\mathbb{T}\,e^{ij_a\hat\phi^a_H}=\mathbb{T}\,\exp\Big(i\int_{t_-}^{t_+}dt\,j(t)\,\hat\phi_H(t)\Big)=\mathbb{T}\,\exp\Big(i\int_{t_-}^{t_+}dt\,\big(J_i(t)\,\hat q^i_H(t)+I^i(t)\,\hat p^H_i(t)\big)\Big), \tag{7.11}$$

where similarly to (6.63) $j_a(t)=J_i(t),I^i(t)$ are the sources dual respectively to the operators of phase space coordinates and momenta.

As in (6.66) this object as a functional of the c-number source $j(t)$, $Z(t_+,t_-)=Z(t_+,t_-)[\,J(t),I(t)\,]$ generates chronological products of Heisenberg operators,

$$\frac{1}{i^n}\frac{\delta^n\hat Z(t_+,t_-)}{\delta j_{a_1}...\delta j_{a_n}}=\mathbb{T}\,\big(\hat\phi^{a_1}_H...\hat\phi^{a_n}_H\,\hat Z(t_+,t_-)\big), \tag{7.12}$$

every variational derivative lowers from the exponential the relevant operator and places it in chronological order with respect to all other operators, including those contained in $\hat Z(t_+,t_-)$. Therefore, for a generic operator functional $F[\,\hat\phi_H\,]$ one has

$$\mathbb{T}\,\big(\,F[\,\hat\phi_H\,]\hat Z[\,j\,]\,\big)=F\Big[\,\frac{\delta}{i\,\delta j}\,\Big]\,\hat Z[\,j\,]. \tag{7.13}$$

Another obvious property of $\hat Z(t_+,t_-)$ is that it satisfies with respect to $t_\pm$ the "left/right Schroedinger" equations with the Hamiltonians $\mp i\,\hat\phi_H(t_\pm)j(t_\pm)$,

$$i\partial_{t_+}\hat Z(t_+,t_-)=-j(t_+)\hat\phi_H(t_+)\,\hat Z(t_+,t_-),\quad i\partial_{t_-}\hat Z(t_+,t_-)=\hat Z(t_+,t_-)\,j(t_-)\hat\phi_H(t_-). \tag{7.14}$$

Consider now $\hat Z=\hat Z(t_+,t_-)$ at $t_\pm\to\pm\infty$ and apply to it the operation $\partial_t\big(\delta/\delta j_a(t)\big)$,

$$\partial_t\Big(\frac{\delta}{i\,\delta j_a(t)}\Big)\,\hat Z=\partial_t\mathbb{T}\,\big(\,\hat\phi^a_H(t)\,\hat Z\,\big)=\partial_t\Big(\hat Z(\infty,t)\,\hat\phi^a_H(t)\,\hat Z(t,-\infty)\Big). \tag{7.15}$$

Then consecutively use Eq.(7.14), the Heisenberg equation of motion

$$\partial_t\hat\phi_H=\frac{1}{i}\,[\,\hat\phi_H,\hat H\,]=\widehat{\{\phi,H\}}\,\big|_{\,\phi\to\hat\phi_H}\,, \tag{7.16}$$

and the relation (7.13) to obtain

$$\begin{aligned}\dot\phi^a_H(t)\,\big|_{\,\phi=\delta/i\delta j}\,\hat Z&=\hat Z(\infty,t)\,\Big(i\,[\,\hat\phi^b_H(t),\hat\phi^a_H(t)\,]\,j_b(t)+\widehat{\{\phi^a,H\}}\,\big|_{\,\phi\to\hat\phi_H(t)}\Big)\,\hat Z(t,-\infty)\\&=\mathbb{T}\Big\{\,\big(-\epsilon^{ab}j_b(t)+\widehat{\{\phi^a,H\}}\,\big|_{\,\phi\to\hat\phi_H(t)}\big)\,\hat Z\Big\}\\&=\Big(-\epsilon^{ab}j_b(t)+\{\phi^a,H\}\,\big|_{\,\phi=\delta/i\delta j(t)}\Big)\,\hat Z.\end{aligned} \tag{7.17}$$

Here the symplectic matrix ϵ^{ab} determines canonical equal-time commutator of Heisenberg operators

$$[\,\hat\phi^a_H(t),\hat\phi^b_H(t)\,] = i\,\{\phi^a,\phi^b\} = i\,\epsilon^{ab},\quad \epsilon^{ab} = \begin{bmatrix} 0 & 1\\ -1 & 0\end{bmatrix},\quad \epsilon_{ab}\equiv(\epsilon^{ab})^{-1} = \begin{bmatrix} 0 & -1\\ 1 & 0\end{bmatrix}, \tag{7.18}$$

and it is understood as acting in the space of phase space columns (we omit for brevity the indices of q^i and p_i)

$$\phi = \begin{bmatrix} q\\ p\end{bmatrix}. \tag{7.19}$$

Now the functional variational equation on $\hat Z[\,j\,]$ can be rewritten as

$$\epsilon_{ab}\,\big(\,\dot\phi^b - \{\phi^b,H\}\,\big)\,\big|_{\,\phi=\delta/i\delta j(t)}\,\hat Z[\,j\,] = -j_a(t)\,\hat Z[\,j\,], \tag{7.20}$$

but what we have in the left hand side is just the variational derivative of the canonical action

$$S[\,\phi\,] = S[\,q,p\,] = \int_{-\infty}^{\infty} dt\,\big(p\,\dot q - H(q,p)\big) + B, \tag{7.21}$$

$$\frac{\delta S}{\delta\phi^a} = \epsilon_{ab}\,\big(\,\dot\phi^b - \{\phi^b,H\}\,\big), \tag{7.22}$$

where B is some surface term at the spacetime boundaries $t_\pm\to\pm\infty$, which should guarantee correctness of the variational procedure and which we shall consider in more detail later. Thus finally we have

$$\left[\frac{\delta S[\,\phi\,]}{\delta\phi^a} + j_a\right]\Bigg|_{\,\phi=\delta/i\delta j}\,\hat Z[\,j\,] = 0. \tag{7.23}$$

This is the final form of the *canonical* (or Hamiltonian) Schwinger-Dyson equations. Note that this is a canonical first-order formalism with time derivatives of maximum first order. This is in contrast to the usually considered Lagrangian Schwinger-Dyson equations which are basically equivalent to the the canonical ones and look the same, except that the set of ϕ^a includes only the configuration space coordinates q, and $S[\,q\,]$ is just the Lagrangian action. Later we will discuss this in more detail.

In the next lecture we will solve these Schwinger-Dyson equations in terms of the phase space path integral and will derive the so-called reduction formulae relating the path integral to S-matrix.

7.2 Solution of Schwinger-Dyson equations and canonical path integral

It is useful to go over from the operator $\hat Z[\,j\,]$ to its c-number analogue – its matrix element between the in and out vacuum states defined by Eq.(7.8),

$$Z[\,j\,] = \langle\,0,-\,|\,\hat Z(+\infty,-\infty)\,|\,0,+\,\rangle = \langle\,0,-\,|\,\mathbb{T}\,e^{i\,j_a\hat\phi^a_H}\,|\,0,+\,\rangle, \tag{7.24}$$

where for brevity we used spacetime condensed notation, $j_a\hat\phi^a_H = \int_{-\infty}^{+\infty}dt\,j_a(t)\,\hat\phi^a_H(t)$, in which the index a carries both space and time continuous labels, and contraction of such indices implies spacetime integration.[2]

In view of (7.8)-(7.9) this vacuum-to-vacuum transition amplitude equals the expectation value of the S-matrix $\hat S[\,j\,]$ in the interaction picture vacuum state $|\,0\,\rangle$,

$$Z[\,j\,] = \langle\,0\,|\,\hat S[\,j\,]\,|\,0\,\rangle, \tag{7.25}$$

where the S-matrix in the presence of the external c-number source j_a is

$$\hat S[\,j\,] = \mathbb{T}\,\big(\,e^{i\,j_a\hat\phi^a_I}\,\hat S\,\big) = \mathbb{T}\,\exp\Big[-i\int_{-\infty}^{+\infty} dt\,\big(\hat V_I(t) - j_a(t)\,\hat\phi^a_I(t)\big)\Big]. \tag{7.26}$$

[2]In what follows we will, when it is necessary for clarity, distinguish between the *spacetime* condensed index a of any variable $\mathcal{O}^a$ and the *canonical* index of $\mathcal{O}^a(t)$ by a simple rule – if the time argument of the variable is explicitly written down then its index is canonical, whereas for spacetime condensed indices and their contractions the time arguments will not be explicitly written down.

In view of (6.66) this is a generating functional of Green's functions – vacuum expectation values of chronological products of interaction picture operators with S-matrix $\hat{S}[\,j\,]$ or equivalently the matrix elements of products of Heisenberg operators between in and out vacua,

$$\frac{1}{i^n}\frac{\delta^n Z[\,j\,]}{\delta j_{a_1}...\delta j_{a_n}} = \langle\,0\,|\,\mathbb{T}\big(\hat{\phi}_I^{a_1}...\hat{\phi}_I^{a_n}\,\hat{S}[\,j\,]\big)\,|\,0\,\rangle = \langle\,0,-\,|\,\mathbb{T}\big(\hat{\phi}_H^{a_1}...\hat{\phi}_H^{a_n}\big)\,|\,0,+\,\rangle. \tag{7.27}$$

At $j=0$ the Heisenberg operators $\hat{\phi}_H(t)$ obviously coincide with those defined in Eqs.(7.6)-(7.7), while for the nonvanishing sources – the situation which is called *off shell* – they are determined by the unitary evolution with the total *Schroedinger picture* Hamiltonian including the source term,

$$\hat{H} \to \hat{H}_{[\,j\,]}(\hat{\phi},t) = \hat{H}(\hat{\phi}) - j_a(t)\,\hat{\phi}^a. \tag{7.28}$$

Note that in this case even in the Schroedinger picture the Hamiltonian $\hat{H}_{[\,j\,]}(\hat{\phi},t)$ is explicitly time dependent, so that in Eq.(7.7) the following replacement should be done

$$e^{i\hat{H}t} \to \mathbb{T}\,\exp\Big[\,i\int_0^t dt'\,\hat{H}_{[\,j\,]}(\hat{\phi},t')\,\Big]. \tag{7.29}$$

In what follows, for brevity, we will not mark off-shell Heisenberg operators by extra label.

Problem 7.1. Show that Eqs.(7.5) and (7.7) hold off shell with $\hat{\Omega}(t)$ replaced by

$$\hat{\Omega}_{[\,j\,]}(t) = \mathbb{T}^\dagger\left\{\exp\Big[\,i\int_0^t dt'\,\hat{H}_{[\,j\,]}(\hat{\phi},t')\Big]\right\}\,e^{-i\hat{H}_0 t}, \tag{7.30}$$

where $\mathbb{T}^\dagger$ denotes the anti-chronological ordering. Then the off-shell operator of unitary evolution and off-shell S-matrix read

$$\hat{U}_{[\,j\,]}(t_+,t_-) \equiv e^{i\hat{H}_0 t}\,\mathbb{T}\left\{\exp\Big[\,i\int_{t_-}^{t_+} dt\,\hat{H}_{[\,j\,]}(\hat{\phi},t)\Big]\right\}\,e^{-i\hat{H}_0 t} = \hat{\Omega}^\dagger_{[\,j\,]}(t_+)\hat{\Omega}_{[\,j\,]}(t_-), \tag{7.31}$$

$$\hat{S}[\,j\,] = \hat{\Omega}^\dagger_{[\,j\,]}(+\infty)\,\hat{\Omega}_{[\,j\,]}(-\infty). \tag{7.32}$$

Altogether the off-shell extension corresponds to including the sources for all phase space coordinates and momenta in the full action of the theory

$$S[\,\phi\,] \to S[\,\phi,j\,] = S[\,\phi\,] + j_a\phi^a = S[\,q,p,J,I\,] + \int_{-\infty}^{+\infty} dt\,\big(J_i(t)\,q^i(t) + I^i(t)\,p_i(t)\big). \tag{7.33}$$

The in-out vacuum-to-vacuum matrix element of the *operator* Schwinger-Dyson equation (7.23) implies that the same equation holds for the generating functional $Z[\,j\,]$ (note that neither $|\,0,\pm\,\rangle$ nor $\hat{\Omega}(\pm\infty)$ in (7.10) depend on the source – these are on-shell objects)

$$\left[\frac{\delta S[\,\phi\,]}{\delta\phi^a} + j_a\right]\bigg|_{\phi=\delta/i\delta j} Z[\,j\,] = 0. \tag{7.34}$$

As equations of motion $\delta S/\delta\phi^a = \epsilon_{ab}\left(\dot{\phi}^b - \{\phi^b,H\}\right)$ involve time derivatives, this equation in partial and variational derivatives requires setting the boundary conditions both in time and in the functional space of $j_a = j_a(t)$. The boundary conditions at $t\to\pm\infty$ follow from the following observation. Consider $\delta Z/i\delta j(t)$ at $t\to-\infty$ (the quantity that stands under the time derivative and, therefore, requiring initial conditions). It equals

$$\frac{\delta Z[\,\phi\,]}{i\delta j(t)}\bigg|_{t\to-\infty} = \langle\,0\,|\,\mathbb{T}\hat{\phi}_I(-\infty)\hat{S}\,|\,0\,\rangle = \langle\,0\,|\,\hat{S}\,\hat{\phi}_I(-\infty)\,|\,0\,\rangle = \langle\,0\,|\,\hat{S}\,\hat{\phi}_I^{(-)}(-\infty)\,|\,0\,\rangle, \tag{7.35}$$

where $\hat{\phi}_I^{(-)}(t) \sim \hat{a}^\dagger\,e^{i\omega t}$ is the negative frequency part of the free theory operator. This means that the positive frequency part of $\delta Z/i\delta j(-\infty)$ equals zero

$$\left[\frac{\delta Z[\,j\,]}{i\,\delta j(t)}\right]^{(+)}_{t\to-\infty} \sim \Big[\,\omega\frac{\delta}{i\,\delta J(t)} + \frac{\delta}{\delta I(t)}\,\Big]\,Z[\,I,J\,]\Big|_{t\to-\infty} = 0, \tag{7.36}$$

where we took into account that this part of $\hat{\phi}_I(t) = \hat{q}_I(t), \hat{p}_I(t)$ in the theory with a free Hamiltonian (6.18) is proportional to the annihilation operator, $\hat{\phi}_I^{(+)}(t) \sim \hat{a} \sim \omega\hat{q}_I(t) + i\hat{p}_I(t)$. Analogous considerations give in the asymptotic future

$$\left[\frac{\delta Z[\,j\,]}{i\,\delta j(t)}\right]^{(-)}_{t\to+\infty} \sim \Big[\,\omega\frac{\delta}{i\,\delta J(t)} - \frac{\delta}{\delta I(t)}\,\Big]\,Z[\,I,J\,]\,\Big|_{t\to+\infty} = 0. \tag{7.37}$$

In other words, only positive and only negative frequency parts of the field are propagated by the unitary evolution respectively to the asymptotic future and asymptotic past of the system. This is expressed by these *homogeneous boundary conditions* in functional derivatives at $t \to \pm\infty$.

Now we will look for the solution of Schwinger-Dyson equation (7.34) in the form of the functional Fourier integral

$$Z[\,j\,] = \int_{\{\phi\}} D\phi\, e^{i\,j_a\phi^a}\,\zeta[\,\phi\,]. \tag{7.38}$$

Note that $\phi = \phi^a$ is a c-number field unlike the operators $\hat{\phi}$ considered above. This is a formal generalization of the multiple Fourier integral to infinite dimensional case with a formally defined Liouville integration measure in phase space of the theory.

$$D\phi = \prod_a d\phi^a = Dq\,Dp = \prod_t\,\Big(\prod_i dq^i(t)\Big)\,\prod_t\,\Big(\prod_i dp_i(t)\Big). \tag{7.39}$$

Integration here runs over the class of fields $\{\,\phi\,\} = \{\,\phi(t)\,\}$ which should be specified regarding the behavior of their "paths" in time at $t \to \pm\infty$. We will derive this class from the boundary conditions for $Z[\,j\,]$ obtained above.

Using (7.38) we consecutively have

$$\begin{aligned}\left[\frac{\delta S[\,\phi\,]}{\delta\phi^a} + j_a\right]\Big|_{\phi=\delta/i\delta j}\,Z[\,j\,] &= \int_{\{\phi\}} D\phi\,\zeta[\,\phi\,]\,\Big(\frac{\delta S[\,\phi\,]}{\delta\phi^a} + j_a\Big)\,e^{i\,j_b\phi^b}\\ &= \int_{\{\phi\}} D\phi\,\zeta[\,\phi\,]\,\Big(\frac{\delta S[\,\phi\,]}{\delta\phi^a} + \frac{\delta}{i\,\delta\phi^a}\Big)\,e^{i\,j_b\phi^b} = \int_{\{\phi\}} D\phi\,\Big(\frac{\delta S[\,\phi\,]}{\delta\phi^a}\,\zeta[\,\phi\,] - \frac{\delta\zeta[\,\phi\,]}{i\,\delta\phi^a}\Big)\,e^{i\,j_b\phi^b},\end{aligned} \tag{7.40}$$

where we functionally integrated by parts on the assumption that the rules of Fourier transform apply also in the functional case. Equating the right hand side to zero we get

$$\frac{\delta\ln\zeta[\,\phi\,]}{i\,\delta\phi^a} = \frac{\delta S[\,\phi\,]}{\delta\phi^a}, \tag{7.41}$$

whence

$$\zeta[\,\phi\,] = \mathrm{const}\,e^{i\,S[\,\phi\,]}, \tag{7.42}$$

where $S[\,\phi\,]$ is the canonical action (7.21) with a yet unknown boundary term S_B. This term should guarantee the correctness of the variational procedure which, in its turn, is determined by the class of functions $\phi(t)$ over which the integration takes place. This class can be derived from the boundary conditions (7.36)-(7.37). Indeed, these boundary conditions imply

$$\Big[\,\omega\frac{\delta}{i\,\delta J(t)} \pm \frac{\delta}{\delta I(t)}\,\Big]\,Z[\,j\,]\,\Big|_{t\to\mp\infty} = \int_{\{\phi\}} D\phi\,e^{i\,j_a\phi^a}\,\zeta[\,\phi\,]\,\big(\,\omega q(t) \pm ip(t)\,\big)\,\Big|_{t\to\mp\infty} = 0, \tag{7.43}$$

and they get granted if this class consists of functions having positive or negative frequency asymptotics respectively at future and past infinities,

$$\{\,\phi(t)\,\}: \quad \omega q(t) \pm ip(t)\,\Big|_{t\to\mp\infty} = 0. \tag{7.44}$$

Thus phase space integration variables in the path integral repeat the boundary conditions for the vacuum-to-vacuum matrix elements of Heisenberg operators.

Then educated guess hints that the surface term in the canonical action takes the form

$$S_B = \frac{i}{2}\,\omega\, q^2\,\Big|_{t\to+\infty} + \frac{i}{2}\,\omega\, q^2\,\Big|_{t\to-\infty}, \tag{7.45}$$

because the variation of the total action (7.21) reads

$$\delta S[\,\phi\,] = \int\limits_{t_-}^{t_+} dt\,\delta\phi^a\epsilon_{ab}\left(\dot\phi^b - \{\phi^b, H\}\right)\Big|_{t_\pm\to\pm\infty} + (p + i\omega\, q)\,\delta q\,\Big|_{t\to+\infty} - (p - i\omega\, q)\,\delta q\,\Big|_{t\to-\infty}, \tag{7.46}$$

and with the boundary conditions of the above type does not at all contain surface terms at $t_\pm = \pm\infty$. It contains only the bulk term which confirms the expression (7.22) for the functional derivative of the action. This expression was critically important for the derivation of the path integral form of $Z[\,j\,]$.

Note that this result fully agrees with quantum mechanical path integral for the kernel of the unitary evolution in the *Schroedinger picture representation*,

$$U(\,t_+, q_+|\,t_-, q_-) = \text{const}\int\limits_{q(t_\pm)=q_\pm} Dq\,Dp\,\exp\Big\{\,i\int_{t_-}^{t_+} dt\,\big(p\dot q - H(q,p)\big)\,\Big\}, \tag{7.47}$$

where the integration runs over the phase space paths interpolating between the coordinate points q_+ at t_+ and q_- at t_-. With the vacuum state in the coordinate representation (6.17) the vacuum-to-vacuum matrix element of this evolution operator equals

$$\begin{aligned}\int dq_+\,dq_-\,\Psi_0^*(\,q_+)\,U(\,t_+, q_+|\,t_-, q_-)\,\Psi_0(\,q_-)&\\ = \text{const}\int Dq\,Dp\,\exp\Big\{\,i\int_{t_-}^{t_+} dt\,\big(p\dot q - H(q,p)\big) - \frac{1}{2}\omega\, q_+^2 - \frac{1}{2}\omega\, q_-^2\,\Big\},&\end{aligned} \tag{7.48}$$

where the path integration now runs also over end points $q_\pm$. The total action in the exponential of this integrand confirms the form of the surface term (7.45). The boundary conditions on a full set of phase space variables $\phi = q, p$ at $t_\pm$ are not directly visible here, but they can be derived at least within the saddle-point approximation for the path integral, which will be done later.

Let us summarize what we have got thus far:

The generating functional of Green's functions is given by the phase space functional integral

$$Z[\,j\,] = \text{const}\int_{\{\phi\}} D\phi\, e^{i\left(S[\,\phi\,] + j_a\phi^a\right)}, \tag{7.49}$$

where $S[\,\phi\,] = S[\,q,p\,]$ is the canonical action on phase space of $\phi^a = q^i(t), p_i(t)$ and integration runs over the class of paths in this space $\{\phi\} = \{\,\phi(t)\,\}$ satisfying special positive-negative frequency boundary conditions

$$S[\,q,p\,] = \int_{-\infty}^{+\infty} dt\,\big(p\dot q - H(q,p)\big) + \frac{i}{2}\,\omega\, q^2\,\Big|_{t\to+\infty} + \frac{i}{2}\,\omega\, q^2\,\Big|_{t\to-\infty}, \tag{7.50}$$

$$\{\,\phi(t)\,\}:\quad \omega q(t) \pm ip(t)\,\big|_{t\to\mp\infty} = 0. \tag{7.51}$$

The constant normalization of $Z[\,j\,]$ remains undefined, but it is physically irrelevant because, as we will see below, it does affect S-matrix.

7.3 Calculation of the path integral via the Gaussian functional integral

The rules of path (functional) integration might seem not rigorous and incomplete for the calculation of S-matrix and relevant physical amplitudes. Here we will show that the rules formulated above are actually sufficient for calculations within perturbation theory. This is based on the reduction of these calculations to the calculation of the Gaussian path integrals. In certain sense this is a counterpart to the mathematical justification of Fourier transform via Gaussian integration.

To begin with, note that the splitting of the total Hamiltonian in the quadratic part and nonlinear interaction $H = H_0 + V$ corresponds to relevant decomposition of the total canonical action into the quadratic and interaction parts (similar to the expansion (6.45)). In supercondensed form this reads as

$$S[\,\phi\,] = \frac{1}{2}\,S_{\phi\phi}\,\phi^2 + S_{\rm int}[\,\phi\,], \tag{7.52}$$

$$S_{\rm int}[\,\phi\,] \equiv -i\int_{-\infty}^{+\infty} dt\,V\big(\phi(t)\big) = \sum_{n=3}^{\infty}\frac{1}{n!}\,S_0^{(n)}\,\phi^n, \tag{7.53}$$

where $S_{\phi\phi} = S^{(2)} \equiv \delta^2 S/\delta\phi\,\delta\phi\,|_{\phi_0}$ and $S_0^{(n)}$ are the second and higher order functional derivatives of the action at the background value of the field ϕ_0 which for simplicity is assumed to be zero. $S_0 = S[\,\phi_0\,]$ is also assumed to be zero (or absorbed into a constant normalization of $Z[\,j\,]$, which we will also omit in what follows), and the linear term is absent, because ϕ_0 is a background solution of equations of motion. With these notations $Z[\,j\,]$ can be rewritten as

$$Z[\,j\,] = \int_{\{\phi\}} D\phi\,e^{i\left(S[\,\phi\,]+j_a\phi^a\right)} = \exp\Big(S_{\rm int}\Big[\,\frac{\delta}{i\delta j}\,\Big]\Big)\int_{\{\phi\}} D\phi\,e^{\frac{i}{2}\,S_{\phi\phi}\phi^2+i\,j\phi}, \tag{7.54}$$

where we extracted the contribution of the interaction term outside of the integral in terms of the exponentiated functional variation operator $\exp\left(S_{\rm int}[\,\delta/i\delta j\,]\right)$. This is, of course, a commutative analogue of the operator relation (7.13), which is based on a simple fact that $e^{i\,j\phi}$ is an eigenfunction of the differentiation operator $\delta/i\delta j$ with the eigenvalue ϕ.

We have got here the Gaussian integral of the form

$$Z_0[\,F,j\,] = \int_{\{\phi\}} D\phi\,e^{\frac{i}{2}\,\phi^a F_{ab}\phi^b + i\,j_a\phi^a}, \tag{7.55}$$

where the functional quadratic form with the kernel $F_{ab} = \delta^2 S/\delta\phi^a\delta\phi^b$ – an operator acting in the space of ϕ^b – reads in spacetime condensed notations and canonical condensed notations respectively as

$$\frac{i}{2}\,\phi^a F_{ab}\phi^b = \frac{i}{2}\int_{t_-}^{t_+} dt\,dt'\,\phi^a(t)\,\frac{\delta^2 S[\,\phi\,]}{\delta\phi^a(t)\,\delta\phi^b(t')}\,\phi^a(t'). \tag{7.56}$$

For a local Hamiltonian action (7.50) F_{ab} is a delta-function type kernel of the first order differential in time operator $S_{ab}(d/dt)$

$$F_{ab} = \frac{\delta^2 S[\,\phi\,]}{\delta\phi^a(t)\,\delta\phi^b(t')} = S_{ab}\Big(\,\frac{d}{dt}\,\Big)\,\delta(t-t'), \tag{7.57}$$

$$S_{ab}\Big(\,\frac{d}{dt}\,\Big) = \epsilon_{ab}\,\frac{d}{dt} - \frac{\partial^2 H(\phi)}{\partial\phi^a\,\partial\phi^b}. \tag{7.58}$$

Problem 7.2. Derive this expression.

Integration in the Gaussian integral (7.55) runs over paths denoted by $\{\phi\}$ with some *linear homogeneous* boundary conditions at $t_\pm$, which in our case are given by positive-negative frequency boundary conditions (7.44). In the calculation of (7.55) their type will be unimportant for a time being, because we want to calculate this integral in a more general case. Also we can keep the limits $t_\pm$ finite, rather than taking them to infinity.

If (7.55) were a finite dimensional integral, its calculation would be trivial and would reduce to finding the determinant of the finite dimensional matrix F_{ab} and its inverse. In the functional case the derivation is more involved and seriously uses the boundary conditions. First of all, shift the integration variables $\phi = \phi_0 + \varphi$ in such a way that ϕ_0 solves the condition of stationarity of the exponential in (7.55) (the "classical equation of motion" for the full exponentiated action) and demand that φ satisfies the same boundary conditions as $\{\phi\}$ – that is belongs to the same functional space $\{\phi\}$. This means that ϕ_0 satisfies the equation

$$F_{ab}\phi_0^b + j_a = 0 \tag{7.59}$$

and the same *linear homogeneous* boundary condition. Then, if F_{ab} is invertible on the functional space $\{\phi\}$, then

$$\phi_0^a = G^{ab} j_b, \quad F_{ab} G^{bc} = -\delta_a^c, \tag{7.60}$$

where G^{ab} is the inverse of F_{ab} on this space – the Green's function of the operator (7.57) subject the boundary conditions of $\{\phi\}$. Using the decomposition $\phi = \phi_0 + \varphi$ in (7.55) we have

$$Z_0[\,F, j\,] = \exp\Big(\frac{i}{2}\, j_a G^{ab} j_b\Big) \int_{\{\phi\}} D\varphi\, e^{\frac{i}{2}\,\varphi^a F_{ab} \varphi^b} = \exp\Big(\frac{i}{2}\, j_a G^{ab} j_b\Big)\, Z_0[\,F, 0\,]. \tag{7.61}$$

To find $Z_0[\,F, 0\,]$ make the following chain of identical transformations for the variation of this quantity with respect to the operator variation, $F_{ab} \to F_{ab} + \delta F_{ab}$,

$$\delta_F Z_0[\,F, 0\,] = \frac{i}{2} \int_{\{\phi\}} D\varphi\, (\varphi\, \delta F \varphi)\, e^{\frac{i}{2}\,\varphi F \varphi} = \frac{i}{2}\, \delta F_{ab}\, \frac{\delta^2 Z_0[\,F, j\,]}{i\delta j_b\, i\delta j_a}\Big|_{\,j=0} = \frac{1}{2}\, \delta F_{ba}\, G^{ab} Z_0[\,F, 0\,], \tag{7.62}$$

whence

$$\delta_F\big(\ln Z_0[\,F, 0\,]\big) = -\frac{1}{2}\, \delta F_{ba} \big(F^{-1}\big)^{ab} = -\frac{1}{2}\, \mathrm{Tr}\,\big(F^{-1} \delta F\big), \tag{7.63}$$

or

$$Z_0[\,F, 0\,] = \mathrm{const}\, \exp\Big(-\frac{1}{2}\, \mathrm{Tr}\, \ln F\Big) = \mathrm{const}\, \big(\mathrm{Det}\, F\,\big)^{-1/2}, \tag{7.64}$$

where Tr and Det denote the *functional trace* and the *functional determinant* of the operator on the corresponding space of functions with given boundary conditions.

Problem 7.3. Prove this formal relation, $\ln \mathrm{Det}\, F = \mathrm{Tr}\, \ln F$, between the trace of the logarithm of a linear operator and its determinant.
Hint: Use variational equation for a determinant and write down the integral representation for $F^{-1} = \int_0^\infty ds\, e^{-sF}$ and for $\ln(F/F_0) = -\int_0^\infty ds\, (e^{-sF} - e^{-sF_0})/s$.

Thus, finally the Gaussian path integral equals

$$Z_0[\,F, j\,] \equiv \int_{\{\phi\}} D\phi\, e^{\frac{i}{2}\,\phi^a F_{ab} \phi^b + i\, j_a \phi^a} = \mathrm{const}\, \Big(\mathrm{Det}\, F_{ab}\Big)^{-1/2} \exp\Big(\frac{i}{2}\, j_a G^{ab} j_b\Big), \quad F_{ab} G^{bc} = -\delta_a^c, \tag{7.65}$$

where the Green's function G^{ab} and the functional determinant of F_{ab} are both determined by boundary conditions of the functional space $\{\phi\}$. For a finite dimensional multiple Gaussian path integral the question of boundary conditions never arises, but in the path integral case such specification regarding boundary conditions is critically important, because they are invoked to fix uniquely G^{ab} and $\mathrm{Det}\, F_{ab}$.

Using this result for the Gaussian path integral in (7.54) we get

$$Z[\,j\,] = \mathrm{const}\, \exp\Big(i\, S_{\mathrm{int}}\Big[\,\frac{\delta}{i\delta j}\,\Big]\Big) \exp\Big(\frac{i}{2}\, j_a G^{ab} j_b\Big), \tag{7.66}$$

where all preexponential factors, including the j-independent $(\mathrm{Det}\, F)^{-1/2}$ (note that $F = S_{\phi\phi}\,|_{\,\phi_0}$ is independent of the source), are included into the constant normalization coefficient.

Lecture 8. S-matrix in Lagrangian formalism and LSZ reduction formulae

- Wick theorem for chronological products
- Path integral in the Lagrangian form – integration over momenta
- Transition to Lagrangian formalism
- Lehman-Symanzik-Zimmermann (LSZ) reduction formulae

8.1 Wick theorem for chronological products

The above expression for the generating functional, which was derived from the Gaussian integration method, should of course be understood as a perturbation theory in interaction part of the action

$$\exp\Big(i\,S_{\rm int}\Big[\,\frac{\delta}{i\delta j}\,\Big]\Big) = 1 + i\,S_{\rm int}\Big[\,\frac{\delta}{i\delta j}\,\Big] + \frac{1}{2}\,\Big(i\,S_{\rm int}\Big[\,\frac{\delta}{i\delta j}\,\Big]\Big)^2 + \dots\,. \tag{8.1}$$

We show now that it corresponds to the functional formulation of the Wick theorem for chronological products – the so-called Chori formula. For this purpose this expression can be further transformed as follows.

We represent the second factor in (7.66) as the result of action of the exponentiated differential operator in variational derivatives $\delta/\delta\phi$ on $e^{iJ\phi}$ at $\phi=0$

$$\exp\Big(\frac{i}{2}\,j_a G^{ab} j_b\Big) = \exp\Big(-\frac{i}{2}\,\frac{\delta}{\delta\phi^a}\,G^{ab}\,\frac{\delta}{\delta\phi^b}\Big)\,e^{ij\phi}\,\Big|_{\,\phi=0}, \tag{8.2}$$

and then commute the differential operators in ϕ with that in j,

$$\begin{aligned} Z[\,j\,] &= {\rm const}\,\exp\Big(i\,S_{\rm int}\Big[\,\frac{\delta}{i\delta j}\,\Big]\Big)\,\exp\Big(-\frac{i}{2}\,\frac{\delta}{\delta\phi^a}\,G^{ab}\,\frac{\delta}{\delta\phi^b}\Big)\,e^{ij\phi}\,\Big|_{\,\phi=0}\,, \\ &= {\rm const}\,\exp\Big(-\frac{i}{2}\,\frac{\delta}{\delta\phi^a}\,G^{ab}\,\frac{\delta}{\delta\phi^b}\Big)\,e^{\,iS_{\rm int}[\,\phi\,]+ij\phi}\,\Big|_{\,\phi=0} \end{aligned} \tag{8.3}$$

This result of the Gaussian integration and perturbation theory in $S_{\rm int}$ corresponds to the transformation of the chronological operator ordering to the normal ordering – Wick theorem for chronological products of interaction picture operators. This is the analogue of the Wick theorem for their symmetrized products. For any monomial of operators $\hat\varphi$ linear in creation-annihilation operators it reads as a sum of normal products of the monomials with all possible *chronological* contractions of pairs of operators $\hat\varphi$,

$$\mathbb{T}\,\hat\varphi_1...\hat\varphi_n =:\hat\varphi_1...\hat\varphi_n:\; + \sum_{\substack{\rm single\\ \rm contractions}} :\underbrace{\hat\varphi_1\hat\varphi_2}...\hat\varphi_n:\; + \sum_{\substack{\rm double\\ \rm contractions}} :\hat\varphi_1\hat\varphi_2\hat\varphi_3\hat\varphi_4...\hat\varphi_n:\; + \dots\,, \tag{8.4}$$

$$\underbrace{\hat\varphi_1\hat\varphi_2} = \mathbb{T}\,(\hat\varphi_1\hat\varphi_2) - :\hat\varphi_1\hat\varphi_2:\; = \langle\,0\,|\,\mathbb{T}\,\hat\varphi_1\hat\varphi_2\,|\,0\,\rangle. \tag{8.5}$$

Similarly to (6.38) for monomials of interaction picture operators it can be rewritten in the functional form with the help of the chronological contraction D_T^{ab}

$$\mathbb{T}\Big(\hat\phi_I^{a_1}...\hat\phi_I^{a_n}\Big) = \;:\,\exp\Big(\frac{1}{2}\,\frac{\delta}{\delta\phi^a}\,D_T^{ab}\,\frac{\delta}{\delta\phi^b}\Big)\,\phi^{a_1}...\phi^{a_n}\,\Big|_{\,\phi\to\hat\phi_I}\,:\,, \tag{8.6}$$

$$D_T^{ab} = \underbrace{\hat\phi_I^a\,\hat\phi_I^b} = \langle\,0\,|\,\mathbb{T}\,\hat\phi_I^a\,\hat\phi_I^b\,|\,0\,\rangle = \theta(t-t')\,\langle\,0\,|\,\hat\phi_I^a(t)\,\hat\phi_I^b(t')\,|\,0\,\rangle + \theta(t'-t)\,\langle\,0\,|\,\hat\phi_I^b(t')\,\hat\phi_I^a(t)\,|\,0\,\rangle \tag{8.7}$$

Here in $D_T^{ab}=D_T^{ab}(t,t')$ the arguments t and t' are associated with the condensed indices a and b respectively.

Let us prove now that this contraction is directly related to the Green's function G^{ab} of the operator (7.57) with the positive/negative frequency boundary conditions (7.44) at future/past infinities,

$$G^{ab} = iD_T^{ab} = i\,\langle\,0\,|\,\mathbb{T}\,\hat\phi_I^a\,\hat\phi_I^b\,|\,0\,\rangle. \tag{8.8}$$

In view of the positive/negative frequency decomposition (6.41) $D_T^{ab}(t,t')\sim e^{\mp i\omega_A t}$ at $t\to\pm\infty$, so that it satisfies the same boundary conditions as $G^{ab}=G^{ab}(t,t')$. Moreover, by acting with the operator (7.57) on $D_T^{ab}(t,t')$ we get

$$S_{ab}(d/dt)\,D_T^{bc}(t,t') = \delta(t-t')\,\epsilon_{ab}\langle\,0\,|\,[\hat\phi_I^b(t),\hat\phi_I^c(t')]\,|\,0\,\rangle = i\,\delta(t-t')\,\delta_a^c. \tag{8.9}$$

Here we took into account that $S_{ab}\hat\phi_I^b=0$ and used a usual property of the step function, $\partial_t\theta(t-t')=\delta(t-t')$ along with the equal-time commutator of phase space operators and normalization of the vacuum state $\langle\,0\,|\,0\,\rangle = 1$. Thus iD_T^{ab} satisfies the same equation as G^{ab},

$$S_{ab}\,G^{bc} = -\delta_a^c, \tag{8.10}$$

and has the same boundary conditions at $t \to \pm\infty$. As we believe that these boundary conditions uniquely define the Green's function of the operator S_{ab}, the two-point function $i\,D_T^{ab}$ indeed coincides with G^{ab}.[3]

With this observation Wick theorem for chronologically ordered monomials (8.6) can obviously be extended to chronologically ordered functionals expandable in power series

$$\mathbb{T}F[\,\hat{\phi}_I\,] = :\,\exp\left(-\frac{i}{2}\,\frac{\delta}{\delta\phi^a}\,G^{ab}\,\frac{\delta}{\delta\phi^b}\right)\,F[\,\phi\,]\,\Big|_{\phi\to\hat{\phi}_I}:. \tag{8.11}$$

When applied to the functional $F[\,\phi\,] = e^{\,iS_{\rm int}[\,\phi\,]+i\phi\,j}$ – the chronological ordering symbol of the S-matrix in the presence of sources (7.26) (see the notation for the interaction part of the action (7.53))

$$\hat{S}[\,j\,] = \mathbb{T}\,\big(\,e^{\,i\,j_a\hat{\phi}^a_I}\hat{S}\,\big) = \mathbb{T}\,\exp\Big(i\,S_{\rm int}[\,\hat{\phi}_I\,] + i\hat{\phi}_I j\Big), \tag{8.12}$$

and bearing in mind that $\langle\,0\,|\!:\mathcal{O}[\,\phi_I\,]\!:|\,0\,\rangle = \mathcal{O}[\,0\,]$ this chronological Wick theorem immediately gives

$$\langle\,0\,|\,\hat{S}[\,j\,]\,|\,0\,\rangle = \exp\Big(-\frac{i}{2}\,\frac{\delta}{\delta\phi^a}\,G^{ab}\,\frac{\delta}{\delta\phi^b}\Big)\,\exp\Big(i\,S_{\rm int}[\,\phi\,] + i\phi\,j\Big)\,\Big|_{\phi=0}, \tag{8.13}$$

which coincides up to a constant factor with $Z[\,j\,]$ – the generating functional (8.3). This derivation in fact recovers the original definition of the generating functional as a vacuum expectation value of S-matrix in the presence of sources (7.25) and can serve as a confirmation of equivalence between the Gaussian path integration method and the functional formulation of the chronological Wick theorem. Unit value of the normalization constant in (8.3), which is not determined from the derivation of the Gaussian integral in (7.62)-(7.64), easily follows from the fact that $\hat{S} = \hat{1}$ when both the interaction $S_{\rm int}$ and the sources j are switched off.

Now we aready to derive the expression for on-shell S-matrix directly in terms of the generating functional $Z[\,j\,]$ – the so-called reduction formulae. We have for on-shell S-matrix

$$\begin{aligned}\hat{S} = \mathbb{T}\,\exp\Big(i\,S_{\rm int}[\,\hat{\phi}_I\,]\Big) &= :\,\exp\Big(-\frac{i}{2}\,\frac{\delta}{\delta\phi^a}\,G^{ab}\,\frac{\delta}{\delta\phi^b}\Big)\,\exp\Big(i\,S_{\rm int}[\,\phi\,]\Big)\,\Big|_{\phi\to\hat{\phi}_I}:\\ &= :\,\exp\Big(i\,S_{\rm int}\Big[\,\frac{\delta}{i\delta j}\,\Big]\Big)\,\exp\Big(-\frac{i}{2}\,\frac{\delta}{\delta\phi^a}\,G^{ab}\,\frac{\delta}{\delta\phi^b}\Big)\,e^{\,i\,j\phi}\,\Big|_{\phi\to\hat{\phi}_I,\,j=0}:\\ &= :\,\exp\Big(i\,S_{\rm int}\Big[\,\frac{\delta}{i\delta j}\,\Big]\Big)\,\exp\Big(\frac{i}{2}\,j_aG^{ab}j_b + i\,j_a\hat{\phi}^a_I\Big):\,\Big|_{j=0},\end{aligned} \tag{8.14}$$

where we commuted the differential operators in variational derivatives with respect to ϕ and j. Now we can rewrite the exponential as follows

$$\frac{i}{2}\,j_aG^{ab}j_b + i\,j_a\hat{\phi}^a_I = \frac{i}{2}\,(j_a - \hat{\phi}^c_I\,\overrightarrow{S}_{ca})\,G^{ab}(j_b - \overleftarrow{S}_{bd}\,\hat{\phi}^d_I). \tag{8.15}$$

Here $\overrightarrow{S}_{ca}$ denotes the usual action of the differential operator to the right, whereas $\overleftarrow{S}_{bd}$ implies its action to the left in the sense of integration by parts of its time (and space if ever) derivatives. Explicitly, for any two test functions

$$\psi^b\,\overleftarrow{S}_{bd}\,\phi^d = \int\limits_{-\infty}^{+\infty} dt\,\Big(-\frac{d}{dt}\psi^b(t)\epsilon_{bd} - \psi^b(t)H_{bd}\Big)\,\phi^d(t) = \int\limits_{-\infty}^{+\infty} dt\,\big[\,S_{db}(d/dt)\psi^b(t)\,\big]\,\phi^d(t), \tag{8.16}$$

where $H_{bd} = \partial^2 H(\phi)/\partial\phi^b\,\partial\phi^d$ and in the second equality we use the symmetry of the operator (7.57), $S_{bd} = S_{db}$. It follows from the symmetry of second order functional derivatives or from the symmetry properties under the functional transposition $(d/dt)^T = -d/dt$ and $\epsilon_{bd} = -\epsilon_{db}$. Obviously, the quadratic forms differing by the

[3] We agreed above to distinguish spacetime condensed and canonical condensed notations by either omitting or explicitly writing down time labels. Correspondingly for differential in time operators and their kernels when the time derivative is explicitly written down as an argument of the operator, its indices are assumed to be of *canonical condensed* nature. For example, the chain of notations for one and the same object, $S_{\phi\phi}(\equiv\,\delta^2 S/\delta\phi\,\delta\phi) = S_{ab}(\equiv\,\delta^2 S/\delta\phi^a\delta\phi^b) = S_{ab}(d/dt)\,\delta(t-t')$, runs from the supercondensed notation to spacetime condensed and then canonical condensed one. Correspondingly, the action of the operator in these notations looks like $S_{\phi\phi}\phi = S_{ab}\phi^b = \int dt'\,S_{ab}(d/dt)\,\delta(t-t')\,\phi^b(t') = S_{ab}(d/dt)\,\phi^b(t)$.

direction of the operator action in their kernels differ from each other by the boundary term following from integration by parts. In case of the first-order differential in time operator S_{ab} this reads

$$\psi^a \overrightarrow{S}_{ab}\,\phi^b = \psi^a \overleftarrow{S}_{ab}\,\phi^b + \psi^a(t)\epsilon_{ab}\phi^b(t)\,\Big|_{t=-\infty}^{t=+\infty}, \tag{8.17}$$

With these notations the right hand side of (8.15) indeed reproduces the left hand side, because

$$\overrightarrow{S}_{\phi\phi}\,G = G\,\overleftarrow{S}_{\phi\phi} = -1, \quad \overrightarrow{S}_{\phi\phi}\,\hat\phi_I = \hat\phi_I\,\overleftarrow{S}_{\phi\phi} = 0, \tag{8.18}$$

and the S-matrix in Eq.(8.14) takes the form

$$\hat S\; = :\; \exp\Big(i\,S_{\rm int}\Big[\,\frac{\delta}{i\delta j}\,\Big]\Big)\;\exp\Big(\frac{i}{2}\,j_a G^{ab} j_b\Big)\,\Big|_{j=-\hat\phi_I \overrightarrow{S}_{\phi\phi}} :\,, \tag{8.19}$$

but what stands here under the sign of normal ordering is just the generating functional $Z[\,j\,]$ (see Eq.(7.66)) at a special value of the source. Therefore

$$\hat S\; = : Z\big[-\hat\phi_I\,\overrightarrow{S}_{\phi\phi}\,\big] :\;. \tag{8.20}$$

The normal ordering symbol of S-matrix is the generating functional of Green's functions $Z[\,j\,]$ at *weakly vanishing* value of the source expressed in terms of the interaction picture field ϕ_I

$$j_a = -\phi_I^b\,\overrightarrow{S}_{ba}\;. \tag{8.21}$$

Weakly vanishing or *on-shell* source means that the contribution of the source reduces to surface terms, because integration by parts of the operator $\overrightarrow{S}_{ba}$ reverses its action on the interaction picture field ϕ_I and annihilates it along with its local contribution in the bulk. For example, if the generating functional is represented in terms of the path integral (7.49),

$$\begin{aligned} Z\big[-\phi_I\,\overrightarrow{S}_{\phi\phi}\,\big] &= \text{const}\int_{\{\phi\}} D\phi\,\exp\Big(iS[\,\phi\,] - i\phi_I^b\,\overrightarrow{S}_{ba}\,\phi^a\Big)\\ &= \text{const}\int_{\{\phi\}} D\phi\,\exp\Big(i\,S[\,\phi\,] - i\,\phi_I^b\epsilon_{ba}\phi^a\,\big|_{-\infty}^{+\infty}\Big), \end{aligned} \tag{8.22}$$

then what remains from the source term is just a symplectic form bilinear in $\phi_I(t)$ and the integration field $\phi(t)$ located at $t\to\pm\infty$.

8.2 Path integral in the Lagrangian form – integration over momenta

For theories with the Lagrangian action quadratic in velocities $\dot q$ and the corresponding Hamiltonians quadratic in canonical momenta p,

$$S_L[\,q\,] = \int dt\,\Big[\,\frac{1}{2}\,a_{ik}(q)\,\dot q^i\dot q^k + b_i(q)\,\dot q^i - V(q)\,\Big], \tag{8.23}$$

$$p_i = a_{ik}(q)\,\dot q^k + b_i \equiv p_i^0(q,\dot q), \tag{8.24}$$

$$H(q,p) = \frac{1}{2}\,p_i a^{ik} p_k - p_i a^{ik} b_k + V(q) + \frac{1}{2}\,b_i a^{ik} b_k, \quad a^{ik} = \big(a_{ik}\big)^{-1}, \tag{8.25}$$

the above formalism can be easily reformulated in terms of the Lagrangian formalism without resorting to the Hamiltonian framework. This can be done by explicit integration in the canonical path integral over the momenta. This is possible because this integral is exactly Gaussian one. We have

$$\int Dq\,Dp\,e^{iS[\,q,p\,]} = \int Dq\,\big(\,\text{Det}\,a_{ik}\,\big)^{1/2}\,e^{\,iS_L[\,q\,]}, \tag{8.26}$$

where the Lagrangian action (the subscript L indicating that this is the Lagrangian action) follows from the canonical action by the substitution for the momenta their expression (8.24) in terms of q and $\dot q$,

$$S_L[\,q\,] = S[\,q,p\,]\,\big|_{\,p=p^0(q,\dot q)}\,. \tag{8.27}$$

As the result we also get additional factor in the integration measure which is determined by the functional determinant of the matrix a_{ik} – the kernel of the kinetic quadratic form in the Lagrangian. For local theories this matrix is ultralocal in time, that is proportional to undifferentiated delta function, $a_{ik} = a_{ik}(t)\,\delta(t-t')$, with the spacetime condensed indices $i \mapsto (i,t)$, $k \mapsto (k,t')$. Its functional determinant reads

$$\begin{aligned}\operatorname{Det} a_{ik} = \exp\big(\operatorname{Tr}\ln a_{ik}\big) &= \exp\Big(\int dt\,\big(\operatorname{tr}\ln a_{ik}(t)\big)\,\delta(t-t')\,\big|_{\,t'=t}\Big)\\ &= \exp\Big(\delta(0)\int dt\,\ln\big(\det a_{ik}(t)\big)\Big),\end{aligned} \tag{8.28}$$

where tr and det denote the functional trace and determinant operations with respect to *canonical* condensed indices. Because of $\delta(0)$ this is a pure divergence whose origin can be traced back to the formal definition of path integral with the measure (7.39). Within this definition the measure of integration in the Lagrangian path integral can be understood as

$$Dq\,\big(\operatorname{Det} a\big)^{1/2} = \prod_t dq(t)\,\big(\det a_{ik}(t)\big)^{1/2}, \tag{8.29}$$

where the formal product over the time points is approximated by the skeletonization of the range of the time variable – forming the lattice of points separated by the intervals of length $\Delta t \to 0$. This length becomes a regularization parameter in terms of which the regulated coincidence limit of the delta function is $\delta(0) = 1/\Delta t \to \infty$. Therefore the regularized functional determinant of the ultralocal measure matrix becomes

$$\begin{aligned}\big(\operatorname{Det} a\big)^{1/2} &= \exp\Big(\frac{1}{2}\,\delta(0)\int dt\,\ln\det a_{ik}(t)\Big)\\ &= \exp\Big(\frac{1}{2\Delta t}\sum_t \Delta t\,\ln\det a_{ik}(t)\Big) = \prod_t \big(\det a_{ik}(t)\big)^{1/2},\end{aligned} \tag{8.30}$$

and corresponds to the measure (8.29).

Note that this measure has a form of Riemannian measure in the configuration space of q^i-variables with the metric a_{ik}. It is obvious that under the diffeomorphisms in this functional space $q \to q' = q'(q)$ the matrix a_{ik} transforms as a metric,

$$a_{ik}(q) \to a_{i'k'}(q') = \frac{\delta q^i}{\delta q^{i'}}\,\frac{\delta q^k}{\delta q^{k'}}\,a_{ik}(q), \tag{8.31}$$

if the action $S_L[\,q\,]$ is demanded to behave as a scalar, $S'_L[\,q'\,] = S_L[\,q\,]$. Then the total measure (8.29) remains invariant in view of the transformation $Dq = Dq'\,D(q)/D(q')$ with the functional Jacobian $D(q)/D(q') = \operatorname{Det}(\delta q/\delta q')$. This is a counterpart to the canonical transformations on phase space which leave invariant the Liouville measure $Dq\,Dp$ and the canonical action $S[\,q,p\,]$.

Problem 8.1. Prove Eq.(8.31).

8.3 Transition to Lagrangian formalism

Let us now use Eq.(8.26) to convert the expressions for the generating functional of the Green's functions and reduction formulae for the S-matrix to the Lagrangian form. For this purpose we switch off the sources dual to the phase space momenta, $I^i = 0$, and perform the integration over momenta in the path integral for the generating functional. For theories with Hamiltonians quadratic in momenta this can be done explicitly according to Eqs.(8.26)-(8.27). The result consists in the replacement of the canonical action by the Lagrangian one and integrating only over configuration space coordinates with extra local measure (8.29), which can be interpreted as an additional $\delta(0)$-type contribution to the action

$$S[\,q,p\,] \to S_L[\,q\,] - \frac{i}{2}\,\delta(0)\int dt\,\ln\det a_{ik}(t). \tag{8.32}$$

Explicit applications show that this $\delta(0)$-type contribution only cancels other so-called *volume* divergences arising in perturbation theory calculations and does not explicitly lead to to physical results. Moreover, there exists dimensional regularization of UV divergences which explicitly puts $\delta(0)$ to zero, which allows one to avoid this cancellation explicitly. For this reason we will not focus on the effects of the local measure and systematically disregard it in what follows.

Modulo the contribution of the local measure, integration over canonical momenta in (7.49) leads to the replacement of these momenta by their Lagrangian expressions, $p \to p_0(q,\dot q)$, $\{\phi\} \to \{q\}$. In particular, the configuration space of q-integration becomes the class of the configurations $\{q\}$ subject to positive/negative frequency conditions at future/past infinity, $\{\phi\} \to \{q\}$, defined by this replacement in (7.51). This would lead to the recovery of all the above formalism (8.18)-(8.22) by the replacement of all the canonical phase-space ingredients with their Lagrangian counterparts,

$$\phi = (q,p) \to q, \quad j = (J,I) \to J, \quad S_{\phi\phi} = \frac{\delta^2 S[\,\phi\,]}{\delta\phi\,\delta\phi} \to S_{L,qq} = \frac{\delta^2 S_L[\,q\,]}{\delta q\,\delta q}, \quad G^{\phi\phi} \to G^{qq}, \tag{8.33}$$

$$Z[\,j\,] \to Z[\,J\,] \equiv Z[\,J,0\,] = \text{const}\int_{\{q\}} Dq\, e^{i(S[\,q\,]+J_i q^i)}, \tag{8.34}$$

However, for this to be true we have to provide two important properties.

Firstly, the 2×2 block form of the canonical 2-point Hessian function S_{ab} and its inverse G^{ab} should lead to the Green's function of the Hessian operator of the Lagrangian action $S_{L,qq} = \delta^2 S_L[\,q\,]/\delta q\,\delta q$, $S_{L,qq}G^{qq} = -1$. This is the operator of linear Lagrangian wave equations, $S_{L,qq}q_I = 0$, for c-number free fields q_I or the corresponding free interaction picture Heisenberg operators $\hat q_I$. This property of the Lagrangian formalism can be directly derived from its canonical counterpart.

Problem 8.2. Prove the above property.

Hint 1. Use the block-matrix structure of canonical equations for G^{ab} and ϕ^a_I

$$S_{\phi\phi}\,G^{\phi\phi} = \begin{bmatrix} S_{qq} & S_{qp} \\ S_{pq} & S_{pp} \end{bmatrix}\begin{bmatrix} G^{qq} & G^{qp} \\ G^{pq} & G^{pp} \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad S_{\phi\phi}\,\phi_I = \begin{bmatrix} S_{qq} & S_{qp} \\ S_{pq} & S_{pp} \end{bmatrix}\begin{bmatrix} q_I \\ p_I \end{bmatrix} = 0, \tag{8.35}$$

where various blocks of these matrices obviously denote second-order functional derivatives of the *canonical* action, $S_{qq} = \delta^2 S/\delta q\delta q$, etc. Show then that $S_{L,qq}\,G^{qq} = -1, S_{L,qq}\,q_I = 0$, where $S_{L,qq} = S_{qq} - S_{qp}\,S^{-1}_{pp}\,S_{pq}$. Note that $S_{pp} = \delta^2 S/\delta p_i(t)\delta p_k(t') = -a^{ik}(t)\,\delta(t-t')$ is an ultralocal in time functional matrix, and its inverse S^{-1}_{pp} does not require specification of boundary conditions.

Hint 2. Show, on the other hand, that this operator coincides with the Hessian of the Lagrangian action in view of

$$\frac{\delta^2 S_L[\,q\,]}{\delta q\,\delta q} = \frac{\delta^2}{\delta q\,\delta q}\,S[\,q, p_0(q,\dot q)\,] = S_{qq} + S_{qp}\,\frac{\delta p_0}{\delta q} = S_{qq} - S_{qp}\,S^{-1}_{pp}\,S_{pq}, \tag{8.36}$$

where we took into account that $\delta p_0/\delta q = -S^{-1}_{pp}S_{pq}$. Note that, because $S_{qp} = -\partial_t + ...$ and $S_{pq} = +\partial_t + ...$ are first-order differential operators in time, this is the second-order differential operator, as it should be in the Lagrangian formalism in contrast to first order canonical formalism (cf. Eq.(7.58)),

$$S_{L,qq} = -\frac{d}{dt}\,a_{ik}(t)\,\frac{d}{dt} + ... \Longleftrightarrow S_{\phi\phi} = \epsilon_{ab}\,\frac{d}{dt} + ... \tag{8.37}$$

Secondly, the effect of the weakly vanishing *canonical* source (8.21) – that is the contribution of the symplectic form in (8.22) – should be equally well induced by the weakly vanishing source of the *Lagrangian* formalism

$$J = -q_I\,\overrightarrow{S}_{L,qq}\,. \tag{8.38}$$

This can indeed be easily proven by using in Eq.(8.17) with $\psi=\phi_I$ and with the Lagrangian value $p=p_0(q,\dot q)$ of the momentum in the set of variables $\phi=(q,p)$. We have for the symplectic form in the exponential of (8.22)

$$-\phi_I^a\,\overrightarrow{S}_{ab}\,\phi^b\,\Big|^{t=+\infty}_{t=-\infty,\,p\to p_0}=-\phi_I^a(t)\epsilon_{ab}\phi^b(t)\,\Big|^{t=+\infty}_{t=-\infty}=\Big[\,q_I(t)\,p_0(q,\dot q)-p_I(t)\,q(t)\,\Big]^{t=+\infty}_{t=-\infty}. \tag{8.39}$$

According to boundary conditions on integration variables $q(t)$ at $t\to\pm\infty$ they tend to positive/negative frequency *solutions* of linear equations of motion for free fields. Then, to illustrate the situation in simple theories (8.23) with $a_{ik}=\delta_{ik}$ and $b^i=0$, the Lagrangian value of the momentum is just a time derivative of $q(t)$, $p_0(q,\dot q)=\dot q$, and this symplectic form reduces to

$$\Big[\,q_I(t)\,\dot q(t)-\dot q_I(t)\,q(t)\,\Big]^{t=+\infty}_{t=-\infty}=\int_{-\infty}^{+\infty}dt\,(q_I\,\ddot q-\ddot q_I\,q)=-q_I\big(\overrightarrow{S}_{L,qq}\,q\big)+\big(q_I\,\overleftarrow{S}_{L,qq}\big)q=-q_I\,\overrightarrow{S}_{L,qq}\,q. \tag{8.40}$$

Thererfore in the Lagrangian formalism on-shell reduction of the generating functional is indeed achieved by the weakly vanishing source (8.38) of the Lagrangian formalism.[4]

This finalizes the proof of the fact that the canonical phase space formalism of perturbation theory for S-matrix literally goes over to the Lagrangian formalism in terms of Lagrangian configuration variables q and their Lagrangian action. In what follows, in order to emphasize field-theoretic content of these variables we will denote them by $\varphi^a=\varphi^i(t)$, their spacetime condensed index a including the canonical labels i (discrete spin-tensor indices and spatial coordinates) and time t. The Lagrangian action will be denoted by $S[\,\varphi\,]$ (omitting the L-subscript). With all this let us summarize the obtained results.

8.4 Lehman-Symanzik-Zimmermann (LSZ) reduction formulae

The generating functional $Z[\,J\,]$ of the multiple-point Green's functions – in-out vacuum matrix elements of chronological products of Heisenberg operators or chronological products of interaction picture operators with S-matrix,

$$\frac{\delta^n Z[\,J\,]}{i\delta J_{a_1}...i\delta J_{a_n}}\,\Bigg|_{J=0}=\langle\,0,-\,|\,\mathbb{T}\,\hat\varphi_H^{a_1}...\hat\varphi_H^{a_n}\,|\,0,+\,\rangle=\langle\,0\,|\,\mathbb{T}\big(\hat\varphi_I^{a_1}...\hat\varphi_I^{a_n}\,\hat S\big)\,|\,0\,\rangle, \tag{8.41}$$

has a path integral representation in the form of the integral over field configurations with positive/negative frequency asymptotics at the future/past infinity

$$Z[\,J\,]=\text{const}\int\limits_{\{\varphi^{(\pm)}(\pm\infty)\}}D\varphi\,e^{i\big(S[\,\varphi\,]+J_a\varphi^a\big)}. \tag{8.42}$$

This generating functional has another representation summing up the perturbation series in powers of the interaction part $S_{\rm int}[\,\varphi\,]$ of the total action,

$$Z[\,J\,]=\exp\Big(i\,S_{\rm int}\Big[\,\frac{\delta}{i\delta J}\,\Big]\Big)\,\exp\Big(\frac{i}{2}\,J_aG^{ab}J_b\Big), \tag{8.43}$$

where G^{ab} is the two-point chronological contraction of interaction picture operators, which is the Green's function of the Hessian of the classical action $S_{\varphi\varphi}=\delta^2S/\delta\varphi\delta\varphi$ on empty flat spacetime background

$$G^{ab}=i\,\langle\,0\,|\,\mathbb{T}\,\hat\varphi_I^a\,\hat\varphi_I^b\,|\,0\,\rangle, \tag{8.44}$$

$$\frac{\delta^2S[\,\varphi\,]}{\delta\varphi^a\,\delta\varphi^b}\,\Bigg|_{\varphi_0}G^{bc}=-\delta^c_a, \tag{8.45}$$

[4]For a generic theory the same relation between the canonical and Lagrangian symplectic forms and the Hessian operator can be obtained by introducing the Wronskian operator $W(d/dt)$ obtained by the variation of $p_0(q,\dot q)$ with respect to q, $\delta p_0(t)/\delta q(t')=W(d/dt)\delta(t-t')$. Then for linearized variables $q_1(t)$, $p_1(t)=W(d/dt)q_1(t)$ and $q_2(t)$, $p_2(t)=W(d/dt)q_2(t)$ one has

$$\big[\,q_1(t)\,p_2(t)-q_2(t)\,p_1(t)\,\big]^{+\infty}_{-\infty}=\big[\,q_1(t)\,Wq_2(t)-(Wq_1(t))\,q_2(t)\,\big]^{+\infty}_{-\infty}=\int_{-\infty}^{+\infty}dt\,\Big[-q_1(\overrightarrow{S}_{L,qq}\,q_2)+(q_1\,\overleftarrow{S}_{L,qq})q_2\Big]$$

This Green's function is called Feynman propagator which has positive/negative frequency boundary conditions at $t \to \pm\infty$. Interaction picture operators $\hat{\varphi}_I$ are zero modes of the wave operator $S_{\varphi\varphi}$ and they have a decomposition in complex conjugated positive/negative frequency basis functions $u^a_A/(u^a_A)^*$ with the creation/annihilation operators, $\hat{a}_A/\hat{a}^\dagger_A$, as coefficients,

$$\frac{\delta^2 S[\,\varphi\,]}{\delta\varphi^a\,\delta\varphi^b}\,\bigg|_{\varphi_0}\,\hat{\varphi}^b_I = 0, \quad \hat{\varphi}^a_I = u^a_A \hat{a}_A + u^{a*}_A \hat{a}^\dagger_A, \quad u^a_A \sim e^{-i\omega_A t}, \omega_A > 0. \tag{8.46}$$

S-matrix is given by the normally ordered generating functional $Z[\,J\,]$ with weakly vanishing operator source, whose contribution to the generating functional actually reduces to special boundary terms at $t \to \pm\infty$,

$$\hat{S} = :Z\big[\,-\hat{\varphi}_I\,\overrightarrow{S}_{\varphi\varphi}\,\big]: \;. \tag{8.47}$$

This can be represented as expansion in multiple-point Green's functions acted upon their spacetime entries by the free-theory wave opertor $S_{\varphi\varphi}$ contracted with interaction picture fields,

$$\begin{aligned}\hat{S} &= :\sum_{n=0}^{\infty}\frac{1}{n!}\;\frac{\delta^n Z[\,J\,]}{\delta J_{a_1}...\delta J_{a_n}}\,\bigg|_{J=0}\,(-\overleftarrow{S}_{a_1b_1}\,\hat{\varphi}^{b_1}_I)...(-\overleftarrow{S}_{a_nb_n}\,\hat{\varphi}^{b_n}_I): \\ &=: \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\,\langle\,0,-\,|\,\mathbb{T}\,\hat{\varphi}^{a_1}_H...\hat{\varphi}^{a_n}_H\,|\,0,+\,\rangle\,\big(\overleftarrow{S}_{a_1b_1}\,\hat{\varphi}^{b_1}_I\big)...\big(\overleftarrow{S}_{a_nb_n}\,\hat{\varphi}^{b_n}_I\big): \;. \end{aligned} \tag{8.48}$$

This relation between S-matrix and chronological Green's functions has the name of Lehman-Symanzik-Zimmerman (LSZ) reduction formulae.

The advantage of the Lagrangian version of the path integral, its perturbation theory and relevant LSZ reduction formulae is its Lorentz covariance, because in contrast to the canonical formalism and Dyson T-exponent it does not involve splitting of spacetime into time and space. Manifest Lorentz covariance of the theory, which is hidden behind first principles of quantization (that is canonical commutation relations for phase space operators, etc.), gives enormous advantages in renormalization of ultraviolet divergences and concrete applications. To demonstrate the Lorentz covariant nature of the above expansion for S-matrix, let us show how these terms of look like in simplest theory of massive scalar field

$$\int dx_1...dx_n\,\frac{(-i)^n}{n!}\,\langle\,0,-\,|\,\mathbb{T}\,\hat{\varphi}_H(x_1)...\hat{\varphi}_H(x_n)\,|\,0,+\,\rangle\,\big(\overleftarrow{\Box}_{x_1}-m^2\,\big)...\big(\overleftarrow{\Box}_{x_n}-m^2\,\big)\,:\hat{\varphi}_I(x_1)...\hat{\varphi}_I(x_n):\;, \tag{8.49}$$

where the interaction picture or free scalar field $\hat{\varphi}_I(x)$ satisfies the Lorentz covariant Klein-Gordon equation $(\Box - m^2)\,\hat{\varphi}_I(x) = 0$ with the d'Alembertian operator $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$.

Let us derive general expression for matrix elements of the S-matrix between multi-particle states. The state of m particles can be written in condensed notations as m-th order derivative of the coherent α-state with respect to the c-number α-parameter

$$|\,1,...m\,\rangle = \hat{a}^\dagger_1...\hat{a}^\dagger_m\,|\,0\,\rangle = \frac{\partial}{\partial\alpha_1}...\frac{\partial}{\partial\alpha_m}\,e^{\,\alpha\hat{a}^\dagger}\,|\,0\,\rangle\,\Big|_{\alpha=0}. \tag{8.50}$$

Introduce the generating functional of the matrix elements

$$Z[\,\alpha,\alpha^*\,] = \langle\,0\,|\,e^{\,\alpha^*\hat{a}}\,\hat{S}\,e^{\,\alpha\hat{a}^\dagger}\,|\,0\,\rangle = \langle\,0\,|\,e^{\,\alpha^*\hat{a}}\,:S_N[\,\hat{\varphi}_I\,]:\;e^{\,\alpha\hat{a}^\dagger}\,|\,0\,\rangle, \tag{8.51}$$

$$\langle\,1,...n\,|\,\hat{S}\,|\,1,...m\,\rangle = \frac{\partial}{\partial\alpha^*_1}...\frac{\partial}{\partial\alpha^*_n}\,\frac{\partial}{\partial\alpha_1}...\frac{\partial}{\partial\alpha_m}\,Z[\,\alpha,\alpha^*\,],\,\Big|_{\alpha=0,\alpha^*=0} \tag{8.52}$$

where $S_N[\,\varphi_I\,]$ is the normal symbol of S-matrix, which as we have just derived is the generating functional of Green's functions at weakly vanishing source, see Eq.(8.47),

$$S_N[\,\varphi_I\,] = Z\big[\,-\varphi_I\,\overrightarrow{S}_{\varphi\varphi}\big], \quad \hat{S} =: S_N[\,\hat{\varphi}_I\,]: \;. \tag{8.53}$$

To calculate (8.51) let us use the generalized Wick theorem in the functional form – see Eqs.(6.38)-(6.40) and comments after them for the case of symmetrized products of interaction picture operators. Now instead of normal or chronological ordering we have in (8.51) a special type ordering when all $\hat{a}^\dagger$ stand to the right of $\hat{\varphi}_I$ and $\hat{a}$, all $\hat{\varphi}_I$ are normally ordered with respect to each other and stand to the right of all $\hat{a}$. This ordering, which we will denote by $\mathbb{O}$ can be formalized as

$$\mathbb{O}\left(\hat{\varphi}_I\hat{\varphi}_I\right) =:\hat{\varphi}_I\hat{\varphi}_I:\,, \quad \mathbb{O}\left(\hat{\varphi}_I\hat{a}\right) = \hat{a}\hat{\varphi}_I, \quad \mathbb{O}\left(\hat{\varphi}_I\hat{a}^\dagger\right) = \hat{\varphi}_I\hat{a}^\dagger, \quad \mathbb{O}\left(\hat{a}\hat{a}^\dagger\right) = \hat{a}\hat{a}^\dagger. \tag{8.54}$$

Therefore, if we denote the collection of all operators in Eq.(8.51) by $\hat{\phi} = (\hat{\varphi}_I, \hat{a}, \hat{a}^\dagger)$ then the conversion of the full operator in this equation to the normal ordering can be written down as

$$\begin{aligned} e^{\alpha^*\hat{a}}:S_N[\hat{\varphi}_I]:\,e^{\alpha\hat{a}^\dagger} &\equiv \mathbb{O}\left(e^{\alpha^*\hat{a}}:S_N[\hat{\varphi}_I]:\,e^{\alpha\hat{a}^\dagger}\right)\\ &= \,:\exp\left(\frac{1}{2}\frac{\delta}{\delta\phi}D\frac{\delta}{\delta\phi}\right)e^{\alpha^* a}\,S_N[\varphi_I]\,e^{\alpha a^\dagger}\Big|_{\phi\to\hat{\phi}}:\,, \end{aligned} \tag{8.55}$$

where D is the contraction in the space of all $\hat{\phi}$ given by the difference of $\mathbb{O}$-ordering and normal ordering

$$D = \mathbb{O}\left(\hat{\phi}\,\hat{\phi}\right) - :\hat{\phi}\,\hat{\phi}:\;. \tag{8.56}$$

Problem 8.3. Calculate this contraction and show that

$$Z[\alpha,\alpha^*] = e^{\alpha^*\alpha}\,S_N[\varphi_I(\alpha,\alpha^*)] = e^{\alpha^*\alpha}\,Z[-\varphi_I(\alpha,\alpha^*)\overrightarrow{S}_{\varphi\varphi}], \tag{8.57}$$

$$\varphi_I(\alpha,\alpha^*) = \hat{\varphi}_I\Big|_{\hat{a}^\dagger,\,\hat{a}\,\to\,\alpha^*,\,\alpha} = u\,\alpha + u^*\alpha^*, \tag{8.58}$$

where $\varphi_I(\alpha,\alpha^*)$ is the c-number interaction picture field obtained from the operator $\hat{\varphi}_I$ by the replacement of creation-annihilation operators with the c-number α-parameters.

With this expression for $Z[\alpha,\alpha^*]$ the contributions to transition amplitudes between ingoing and outgoing particle states (8.52) can be classified as follows. Differentiation of $e^{\alpha^*\alpha}$ gives contributions with no scattering

$$\frac{\partial}{\partial\alpha^*_A}\frac{\partial}{\partial\alpha_B}\,e^{\alpha^*\alpha} \Rightarrow \delta_{AB}, \tag{8.59}$$

whereas the differentiation of $S_N[\varphi_I(\alpha,\alpha^*)]$ corresponds to dressing the multiple-point Green's functions with external lines carrying the positive-frequency basis functions u_B for *ingoing* particles and negative-frequency basis functions u^*_A for *outgoing* particles

$$\begin{aligned} &\frac{\partial}{\partial\alpha^*_{A_1}}\cdots\frac{\partial}{\partial\alpha^*_{A_n}}\frac{\partial}{\partial\alpha_{B_1}}\cdots\frac{\partial}{\partial\alpha_{B_m}}\,Z[-\varphi_I(\alpha,\alpha^*)\overrightarrow{S}_{\varphi\varphi}]\\ &\qquad\Rightarrow \frac{(-i)^{n+m}}{(n+m)!}\,(u^*_{A_1}\overrightarrow{S}_{\varphi\varphi})\ldots(u^*_{A_n}\overrightarrow{S}_{\varphi\varphi})\,\langle 0\,|\,\mathbb{T}\,\hat{\varphi}^I_1\ldots\hat{\varphi}^I_{n+m}\,\hat{S}\,|\,0\rangle\,(\overleftarrow{S}_{\varphi\varphi}u_{B_1})\ldots(\overleftarrow{S}_{\varphi\varphi}u_{B_m})\,. \end{aligned} \tag{8.60}$$

Lecture 9. Bose-Fermi systems in the functional formalism of QFT

- Dirac theory and Grassman (fermionic) variables
- Canonical formalism with fermions
- Quantization of boson-fermion systems
- Sources and Gaussian integration over boson-fermion fields

9.1 Dirac theory and Grassman (fermionic) variables

Let us extend the above formalism to systems containing both bosonic and fermionic fields. We begin with the Dirac theory with the action

$$S[\psi] = \int d^4x\,\bar{\psi}\,(i\hat{\partial} - m)\,\psi, \quad \bar{\psi} = \psi^\dagger\gamma^0, \quad \hat{\partial} = \gamma^\mu\partial_\mu \tag{9.1}$$

of the complex 4-component massive spinor field. Here we use the standard representation of Dirac gamma matrices

$$\gamma^\mu\gamma^\nu + \gamma^\mu\gamma^\nu = -2\eta^{\mu\nu}, \quad \eta^{\mu\nu} = \mathrm{diag}\,(-1,+1,+1,+1), \tag{9.2}$$

$$\gamma^0 = \begin{bmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix}. \tag{9.3}$$

Dirac equations for ψ and $\bar\psi$, which follow from the (left and right) variation of the action, have the form

$$\frac{\overrightarrow{\delta}}{\delta\bar\psi}\, S[\,\psi\,] = (i\hat\partial - m)\,\psi = 0, \quad S[\,\psi\,]\,\frac{\overleftarrow{\delta}}{\delta\bar\psi} = -(\,i\partial_\mu\bar\psi\gamma^\mu + m\bar\psi\,) = 0 \tag{9.4}$$

The 4-component column ψ of the spinor field is complex, so that ψ and $\bar\psi$ should be treated as independent fields. In view of their complexity the decomposition of the general solution of the Dirac equation in its basis functions is more complicated than for a real scalar field. It reads along with the momentum space decomposition of the conserved Hamiltonian as follows

$$\psi(x) = \int d^3\mathbf{p}\,\Big(\,u_s^{(-)}(x,\mathbf{p})\,a_s^{(-)}(\mathbf{p}) + u_s^{(+)}(x,\mathbf{p})\,[\,a_s^{(+)}(\mathbf{p})\,]^*\,\Big), \tag{9.5}$$

$$\bar\psi(x) = \int d^3\mathbf{p}\,\Big(\,\overline{u_s^{(+)}(x,\mathbf{p})}\,a_s^{(+)}(\mathbf{p}) + \overline{u_s^{(-)}(x,\mathbf{p})}\,[\,a_s^{(-)}(\mathbf{p})\,]^*\,\Big), \tag{9.6}$$

$$H = i\int d^3\mathbf{x}\,\psi^\dagger\partial_\mu\psi = \int d^3\mathbf{p}\,\omega(\mathbf{p})\,\Big(\,[\,a_s^{(-)}(\mathbf{p})\,]^*\,a_s^{(-)}(\mathbf{p}) - a_s^{(+)}(\mathbf{p})\,[\,a_s^{(+)}(\mathbf{p})\,]^*\,\Big), \tag{9.7}$$

where $u_s^{(\mp)}(x,\mathbf{p}) \sim e^{\mp i\omega(\mathbf{p})t}$, $\omega(\mathbf{p}) > 0$, are positive/negative energy basis functions of the Dirac equation with two $s = \pm 1$ helicities. Note that the negative frequency part of ψ is not complex conjugated to its positive frequency part, but the Hamiltonian is diagonal, $H \sim [\,a^{(\mp)}\,]^*\,a^{(\mp)}$ in what becomes under quantization the creation/annhilation operators, $a^{(\mp)} \to \hat a^{(\mp)}$, $[\,a^{(\mp)}\,]^* \to [\,\hat a^{(\mp)}\,]^\dagger$. However, classically it is not positive-definite, which serves as a motivation for imposing at the quantum level the *anti-commutation* relations. For any two spinor (or fermionic) variables F and G this is the replacement of their commutator by anti-commutator

$$[\,F,G\,]_- \equiv FG - GF \to [\,F,G\,]_+ \equiv FG + GF. \tag{9.8}$$

Then from the positivity of the Hamiltonian and the requirement of the invariance of the theory with respect to the permutation of particles and antiparticles $\hat a^{(+)} \Leftrightarrow \hat a^{(-)}$, $\hat a^{(+)\dagger} \Leftrightarrow \hat a^{(-)\dagger}$,[5] the following commutation relations follow,

$$[\,\hat a_A, \hat a_B^\dagger\,]_+ = \delta_{AB}, \quad A,B \mapsto (\pm, s, \mathbf{p}), \tag{9.9}$$

(all other *anti-commutators* vanish), where we use condensed index to label all modes of the Dirac equation. Therefore we have

$$\langle\,0\,|\,\psi(x)\,\bar\psi(y)\,|\,0\,\rangle = \int d^3\mathbf{p}\,u_s^{(-)}(x,\mathbf{p})\,\overline{u_s^{(-)}(y,\mathbf{p})}\, = (\,i\hat\partial + m\,\hat 1\,)\,i\,G^{(+)}(x,y), \tag{9.10}$$

$$\langle\,0\,|\,\bar\psi(y)\,\psi(x)\,|\,0\,\rangle = \int d^3\mathbf{p}\,u_s^{(-)}(x,\mathbf{p})\,\overline{u_s^{(-)}(y,\mathbf{p})}\, = (\,i\hat\partial + m\,\hat 1\,)\,i\,G^{(-)}(x,y), \tag{9.11}$$

where $\hat 1$ is a unit matrix in the space of ψ and $G^{(\pm)}(x,y)$ are the Wightman functions of the scalar field (6.43)-(6.44).[6] The anti-commutator function then equals

$$[\,\psi(x),\bar\psi(y)\,]_+ = \langle\,0\,|\,\psi(x)\,\bar\psi(y) + \bar\psi(y)\,\psi(x)\,|\,0\,\rangle = i\,(i\hat\partial + m\,\hat 1)\,\tilde G(x,y), \tag{9.12}$$

[5] Superscript labels $(\pm)$ of creation-annihilation operators here distinguish particles from antiparticles related by charge conjugation, rather than the positive/negative frequency splitting.

[6] These relations of summation over polarizations follow from the standard course on the theory of the Dirac field, see *N.N.Bogolyubov and D.V.Shirkov, Quantized Fields, Nauka, Moscow, 1980.*

and expresses in terms of $\tilde{G}(x,y)$ – the scalar field commutator function of Pauli-Jordan from Eq.(6.44). Since $\tilde{G}(x,y)\,|_{x^0=y^0}=0$ and $\partial_0 G(x,y)\,|_{x^0=y^0}=\delta^{(3)}(\mathbf{x}-\mathbf{y})$, the equal time anti-commutator equals

$$[\,\psi(x),\bar{\psi}(y)\,]_+\big|_{x^0=y^0}=\gamma^0\delta^{(3)}(\mathbf{x}-\mathbf{y}). \tag{9.13}$$

This relation suggests that the operator variable $\pi=i\,\bar{\psi}\,\gamma^0$ could be interpreted as an operator of momentum canonically conjugated to ψ and satisfying the canonical *anti-commutation* relation

$$[\,\psi(\mathbf{x}),\pi(\mathbf{y})\,]_+=i\,\delta^{(3)}(\mathbf{x}-\mathbf{y}),\quad [\,\psi(\mathbf{x}),\psi(\mathbf{y})\,]_+=0,\quad [\,\pi(\mathbf{x}),\pi(\mathbf{y})\,]_+=0. \tag{9.14}$$

This interpretation follows from the action of the spinor field (9.1) which is already in the first order formalism – under the (3+1)-decomposition of spacetime it reads

$$S[\,\psi\,]=\int d^4x\,(i\bar{\psi}\,\gamma^0\,\dot{\psi}+i\bar{\psi}\,\gamma^k\partial_k\psi-m\,\bar{\psi}\,\psi\,), \tag{9.15}$$

where the first term is a symplectic form $\pi\dot{\psi}$ and the rest is minus the Hamiltonian of the theory.

9.2 Canonical formalism with fermions

Quantum commutation relations (9.14) lead to reconsidering the status of the spinor theory at the classical level. *Classically* the fields ψ and $\bar{\psi}$ belong to the Grassmann algebra of anti-commuting elements,

$$\begin{aligned}&\psi^I(x)\psi^J(y)=-\psi^J(y)\psi^I(x),\quad \psi^I(x)\bar{\psi}_J(y)=-\bar{\psi}_J(y)\psi^I(x),\quad \bar{\psi}_I(x)\bar{\psi}_J(y)=-\bar{\psi}_J(y)\bar{\psi}_I(x),\\ &\pi_I(x)\psi^J(y)=-\psi^J(y)\pi_I(x),\ \ldots,\end{aligned} \tag{9.16}$$

where we explicitly wrote down the spinor indices of columns of ψ and rows of $\bar{\psi}$. This applies also to the commutation of these fields with their variations

$$\psi^I(x)\delta\psi^J(y)=-\delta\psi^J(y)\psi^I(x),\ \ldots, \tag{9.17}$$

which means that one should define two different (but related) derivatives with respect to Grassmann variables. For any monomial $F=\psi_1\psi_2...\psi_n$ one has the definition of two – right and left – derivatives,

$$\delta F=\delta\psi_1\psi_2...\psi_n+\psi_1\delta\psi_2\psi_3...\psi_n+...=\int dx\,\frac{\delta_R F}{\delta\psi(x)}\delta\psi(x)=\int dx\,\delta\psi(x)\,\frac{\delta_L F}{\delta\psi(x)}. \tag{9.18}$$

Obviously, for even n we have $\delta_R F/\delta\psi(x)=-\delta_L F/\delta\psi(x)$, while for odd n these two derivatives coincide.

More generally we ascribe Grassmannn *parity* to any functional of ψ, $\epsilon(F)\equiv\epsilon_F$ which equals 0 or 1 modulo 2. If F is even in ψ (even element of Grassmannn algebra including purely bosonic quantities with $\epsilon=0$) then $\epsilon_F=0$ and $\epsilon_F=1$ if F is odd in ψ. Then

$$\frac{\delta_R F}{\delta\psi(x)}=-(-1)^{\epsilon(F)}\frac{\delta_L F}{\delta\psi(x)}. \tag{9.19}$$

Another useful notation for left and right derivatives, which we will use, is

$$\frac{\delta_L F}{\delta\psi(x)}=\frac{\overrightarrow{\delta}}{\delta\psi(x)}F,\quad \frac{\delta_R F}{\delta\psi(x)}=F\,\frac{\overleftarrow{\delta}}{\delta\psi(x)}. \tag{9.20}$$

These fermionic functional derivatives allow us to define the Poisson bracket on Grassmann phase space,

$$\{\,F,G\,\}_+=\int d^3\mathbf{x}\,\frac{\delta_R F}{\delta\psi(\mathbf{x})}\,\frac{\delta_L G}{\delta\pi(\mathbf{x})}-(-1)^{\epsilon_F\epsilon_G}\big(\,F\Leftrightarrow G\,\big)=\int d^3\mathbf{x}\,\Big(\frac{\delta_R F}{\delta\psi(\mathbf{x})}\,\frac{\delta_L G}{\delta\pi(\mathbf{x})}+\frac{\delta_R F}{\delta\pi(\mathbf{x})}\,\frac{\delta_L G}{\delta\psi(\mathbf{x})}\Big). \tag{9.21}$$

$$\{\,F,G\,\}_+=-(-1)^{\epsilon_F\epsilon_G}\{\,G,F\,\}_+ \tag{9.22}$$

Problem 9.1. Prove equivalence of these two definitions and this symmetry.

Then Dirac equations can be rewritten as the following canonical fermionic equations of motion

$$\dot{\psi} = \frac{\delta_L H}{\delta\pi} = \{\,\psi, H\,\}_+, \quad \dot{\pi} = -\frac{\delta_R H}{\delta\psi} = \{\,\pi, H\,\}_+, \tag{9.23}$$

$$H = \int d^3\mathbf{x}\,\big(-\pi\,\gamma^0\,\gamma^i\partial_i\psi - im\,\pi\,\gamma^0\,\psi\big) \tag{9.24}$$

Problem 9.2. Derive these equations of motion.

9.3 Quantization of boson-fermion systems

Now we can proceed to the transition from classical to quantum theory of fermionic variables by a usual procedure of canonical quantization. For fermionic observables $F, G, ...$ it consists in the promotion of their fermionic Poisson brackets to anti-commutators, $F, G, ... \to \hat{F}, \hat{G}, ...$, $\{\,F, G\,\}_+ \to \frac{1}{i}\,[\,F, G\,]_+$. Moreover, we can generalize the quantization scheme to general systems including both bosonic and fermionic degrees of freedom. In this general setup, using condensed notations, we have a boson-fermion phase space and various observables with their grassman parities

$$\phi = (q^i, p_i, F(q,p), G(q,p), ...), \quad \epsilon(\phi) = \epsilon_\phi, \quad \epsilon(q^i) = \epsilon(p_i) = \epsilon_i, \quad \phi_1\phi_2 = (-1)^{\epsilon_1\epsilon_2}\phi_2\phi_1. \tag{9.25}$$

Equations of motion

$$\dot{q}^i = \frac{\overrightarrow{\partial}}{\partial p_i}\,H = \{\,q^i, H\,\}_S, \quad \dot{p}_i = -H\,\frac{\overleftarrow{\partial}}{\partial q^i} = \{\,q^i, H\,\}_S, \tag{9.26}$$

can be rewritten in terms of *super-Poisson* bracket defined on combined boson-fermion phase space by the following two equivalent definitions

$$\{\,F, G\,\}_S = F\,\frac{\overleftarrow{\partial}}{\partial q^i}\,\frac{\overrightarrow{\partial}}{\partial p_i}\,G - (-1)^{\epsilon_i}\,F\,\frac{\overleftarrow{\partial}}{\partial p_i}\,\frac{\overrightarrow{\partial}}{\partial q^i}\,G = F\,\frac{\overleftarrow{\partial}}{\partial q^i}\,\frac{\overrightarrow{\partial}}{\partial p_i}\,G - (-1)^{\epsilon_F\epsilon_G}\,G\,\frac{\overleftarrow{\partial}}{\partial q^i}\,\frac{\overrightarrow{\partial}}{\partial p_i}\,F, \tag{9.27}$$

$$\{\,F, G\,\}_S = -(-1)^{\epsilon_F\epsilon_G}\{\,G, F\,\}_S\,. \tag{9.28}$$

Problem 9.3. Prove equivalence of these two definitions and this symmetry.

In this setup quantization proceeds by the usual way,

$$\phi \to \hat{\phi} = (\hat{q}, \hat{p}, \hat{F}, \hat{G}, ...), \quad \{\,F, G\,\}_S \to [\,\hat{F}, \hat{G}\,]_S = \hat{F}\hat{G} - (-1)^{\epsilon_F\epsilon_G}\hat{G}\hat{F} = i\hbar\,\{\,\widehat{F, G}\,\}_S + O(\hbar^2), \tag{9.29}$$

including exact realization of supercommutators of basic phase space operators $[\,\hat{q}, \hat{p}\,] = i\hbar$. Heisenberg equations of motion for Heisenberg operators $\hat{\phi}_H$ remain of course the same, $i\hbar d\hat{\phi}_H/dt = [\,\hat{\phi}_H, \hat{H}\,]$, because of bosonic nature of the Hamiltonian. Therefore, all the formalism of interaction picture representation, generating functional of Green's functions, S-matrix, etc., remain basically the same. We will now only dwell on modifications in this formalism which are necessary in order to take into account inclusion of fermions.

First modification concerns the symmetry properties of multi-particle states (8.50) and definition of chronological products of operators (6.57). Since fermionic creation operators are anti-commuting, $\hat{a}_1^\dagger\hat{a}_2^\dagger = -\hat{a}_2^\dagger\hat{a}_1^\dagger$, changing the order of each pair of those leads to flipping the overall sign. Therefore, each state of fermionic particles is antisymmetric with respect to their permutations. For a state $|\,1, ...m\,\rangle = \hat{a}_1^\dagger...\hat{a}_m^\dagger\,|\,0\,\rangle$ containing both fermions and bosons among the creation operators this property can be formulated in terms of the parity of fermionic permutation. If we denote by $p(1, ...n) = (p_1, ...p_n)$ the permutation of n indices, when each k-th index goes over into p_k, $k \to p_k$. The parity of the p-permutation $P(p)$ equals 0 if this permutation consists of an even number of pair permutations, and it is 1 if the permutation is odd. Then the symmetry of such generic boson-fermion state under this permutation is

$$p\big(|\,1, ...n\,\rangle\big) \equiv |\,p_1, ...p_n\,\rangle = (-1)^{P(p_f(1,...n))}\,|\,1, ...n\,\rangle \tag{9.30}$$

where $p_f(1,...n)$ is a permutation of only fermionic members among $1,2...n$. Similarly, the sign factor should be taken into account in the definition of the chronological product (6.57) and its symmetry property

$$\mathbb{T}\left(\hat{\phi}_1\hat{\phi}_2...\hat{\phi}_n\right) = \sum_{\{p\}}(-1)^{P(p_f(1,...n))}\theta(t_{p_1}-t_{p_2})\theta(t_{p_2}-t_{p_3})...\theta(t_{p_{n-1}}-t_{p_n})\,\hat{\phi}_{p_1}\hat{\phi}_{p_2}...\hat{\phi}_{p_n}, \tag{9.31}$$

$$p\Big(\mathbb{T}\left(\hat{\phi}_1\hat{\phi}_2...\hat{\phi}_n\right)\Big) = (-1)^{P(p_f(1,...n))}\,\mathbb{T}\left(\hat{\phi}_1\hat{\phi}_2...\hat{\phi}_n\right) \tag{9.32}$$

where the indices carry of course all the labels of relevant operators, $\hat{\phi}_k = \hat{\phi}^{a_k}(t_k)$. In particular, for pure Dirac field

$$\mathbb{T}\,\psi(x)\,\bar{\psi}(y) = \theta(x^0-y^0)\,\psi(x)\,\bar{\psi}(y) - \theta(y^0-x^0)\,\bar{\psi}(y)\,\psi(x) = -i\,G_{\rm Dirac}(x,y) \tag{9.33}$$

Problem 9.4. By using (9.10)-(9.13) show that $G_{\rm Dirac}(x,y)$ is the Green's function of Dirac operator – the Hessian of the Dirac action,

$$(\,i\hat{\partial}-m)\,G_{\rm Dirac}(x,y) = -\delta(x-y), \tag{9.34}$$

$$\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}\,S[\,\psi,\bar{\psi}\,]\,\frac{\overleftarrow{\delta}}{\delta\psi(y)} = (\,i\,\hat{\partial}-m\,\hat{1}\,)\,\delta(x-y). \tag{9.35}$$

Now introduce the column notation for the full set of Dirac fields and define the full Hessian of the action,

$$\hat{\phi}^a = \begin{bmatrix} \psi(x) \\ \bar{\psi}(x) \end{bmatrix}, \quad \frac{\overrightarrow{\delta}}{\delta\phi^a}\,S[\,\phi\,]\,\frac{\overleftarrow{\delta}}{\delta\phi^b} = \begin{bmatrix} 0 & i\hat{\partial}+m \\ i\hat{\partial}-m & 0 \end{bmatrix}\delta(x-y), \tag{9.36}$$

and the functional matrix of chronological contractions

$$G^{ab} = i\langle\,0\,|\,\mathbb{T}\,\hat{\phi}^a\hat{\phi}^b\,|\,0\,\rangle = \begin{bmatrix} 0 & i\langle\,0\,|\,\mathbb{T}\,\psi(x)\bar{\psi}(y)\,|\,0\,\rangle \\ i\langle\,0\,|\,\mathbb{T}\,\bar{\psi}(x)\psi(y)\,|\,0\,\rangle & 0 \end{bmatrix}, \tag{9.37}$$

so that finally

$${}_aS_b\,G^{bc} = -\delta^c_a, \qquad {}_aS_b \equiv \frac{\overrightarrow{\delta}}{\delta\phi^a}\,S[\,\phi\,]\,\frac{\overleftarrow{\delta}}{\delta\phi^b}, \tag{9.38}$$

where we introduced a special condensed notation indicating by the position of the subscripts the left and right action of the relevant functional derivatives. This exactly reproduces a similar equation in the bosonic theory – the functional matrix of the chronological contractions is the Green's function of the Hessian of the action. This is the Feynman propagator in the full set of fields with the same choice of positive/negative frequency conditions at spacetime infinity.

Problem 9.5. Prove the following symmetries of the action Hessian and Green's function

$${}_aS_b = (-1)^{\epsilon_a+\epsilon_b+\epsilon_a\epsilon_b}\;{}_bS_a, \quad G^{ab} = (-1)^{\epsilon_a\epsilon_b}G^{ba}. \tag{9.39}$$

9.4 Sources and Gaussian integration over boson-fermion fields

In extending the formalism of generating functional and S-matrix reduction formulae to boson-fermion systems one should be careful how to include the sources J_a dual to fermions φ^a in $Z[\,J\,]$ and how to perform functional differentiation with respect to them. First of all, the sources should obviously be of the same statistics as their fields, $\epsilon(\varphi^a) = \epsilon(J_a)$, and in the exponential of the generating functional their order in $J_a\varphi^a = (-1)^{\epsilon_a}\varphi^a J_a$ should be chosen. Under the left choice for the position of sources in Eq.(7.11) and left functional derivatives in (7.12) we will have for pure fermion case

$$\hat{Z}[\,\bar{\eta}\,] = \mathbb{T}\,\exp\Big(i\int dt\,\bar{\eta}(t)\,\psi(t)\Big), \tag{9.40}$$

$$\mathbb{T}\left(\,\psi_1...\psi_n\,\hat{Z}[\,\bar{\eta}\,]\,\right) = \frac{\overrightarrow{\delta}}{i\delta\bar{\eta}_1}...\frac{\overrightarrow{\delta}}{i\delta\bar{\eta}_n}\,\hat{Z}[\,\bar{\eta}\,], \tag{9.41}$$

and this rule can be extended to the whole functional formalism of bosons and fermions. With this rule the Gaussian path integral (7.66) generalizes to boson-fermion case as (**mind the order of factors in the exponential!**)

$$\int D\varphi\, e^{\,\frac{i}{2}\,\phi^a\,({}_aS_b)\phi^b + i\,J_a\varphi^a} = \text{const}\,\Big(\text{Sdet}\,{}_aS_b\Big)^{-1/2} \exp\Big(\frac{i}{2}\,J_aJ_b\,G^{ba}\Big), \tag{9.42}$$

where the functional *superdeterminant* or *Berezinian* is defined by its variational law in terms of *supertrace* – the sum of the diagonal elements weighted by the parity sign factor,

$$\delta\ln\Big(\text{Sdet}\,K_{ab}\Big) = \big(\,K^{-1}\big)^{ba}\,\delta K_{ab}(-1)^{\epsilon_b} \equiv \text{Str}\,\big(\,K^{-1}\delta K\,\big). \tag{9.43}$$

For a particular case of the block-diagonal matrix K_{ab} consisting of the bosonic block B and fermionic block F this superdeterminant equals

$$K_{ab} = \begin{bmatrix} B & 0 \\ 0 & F \end{bmatrix}, \quad \text{Sdet}\,K = \text{Det}\,B\,\big(\,\text{Det}\,F\,\big)^{-1}. \tag{9.44}$$

Problem 9.6. By the method of Lecture 6 prove Eqs.(9.42)-(9.44).

Lecture 10. Perturbation theory, types of Feynman diagrams and semiclassical expansion

- Factorization of vacuum graphs
- Connected and one-particle irreducible graphs. Effective action
- Semiclassical (loop) expansion and background field formalism
- Effective action in one and two-loop approximations

Perturbation theory for S-matrix and its generating functional give rise to Feynman diagrammatic technique. In particular, Eq.(8.43) or its equivalent form

$$Z[\,J\,] = \exp\Big(-\frac{i}{2}\,\frac{\delta}{\delta\varphi^a}\,G^{ab}\,\frac{\delta}{\delta\varphi^b}\Big)\,\exp\Big(i\,S_{\text{int}}\,[\,\varphi\,] + iJ_a\varphi^a\Big)\Big|_{\varphi=0}, \tag{10.1}$$

Problem 10.1. Derive (10.1) from (8.43).

explicitly show that in the φ-expansion of the exponential of the action (and the source term) all monomials of φ, $\varphi_1...\varphi_n$ should be replaced by the contractions of their pairs. The result is an expansion in powers of the propagator $G^{ab} = G^{IK}(x,y)$, $a = (I,x)$, and sources $J_a = J_I(x)$. Graphically this means that you get the set of terms corresponding to spacetime Feynman graphs. These graphs consist of point vertices connected by lines of two-point propagators, some of these lines ending on the sources J. Vertices correspond to multiple functional derivatives of the action, $S_{a_1...a_n} = \delta^n S_{\text{int}}/\delta\varphi^{a_1}...\delta\varphi^{a_n}$. For local field theories the action is a spacetime integral of the local function of fields and their derivatives to some finite order, so that the structure of the vertex function looks as

$$\frac{\delta^n S_{\text{int}}}{\delta\varphi(x_1)...\delta\varphi(x_n)} = P(x,\partial/\partial x)\,\delta((x_1-x_2)\,\delta(x_1-x_3)...\delta(x_1-x_n), \tag{10.2}$$

where $P(x,\partial/\partial x)$ is some polynomial in derivatives. Therefore, every n-th vertex is associated with just one point which remains after the integration over other $(n-1)$ coordinates is done. The propagators correspond to the Feynman Green's function G^{ab} – the lines connecting the spacetime points associated with condensed indices a and b. The sources sit at the ends of external lines, so that the chronological Green's functions $G(x_1,...x_n) = \langle 0 \,|\, \mathbb{T}\,\hat\varphi(x_1)\hat\varphi...\hat\varphi(x_n)\hat S \,|\, 0 \rangle$ (which are obtained by functional differentiation of these sources) have external propagators terminating at the points $x_1,...x_n$.

Here we will dwell on connectedness properties of Feynman graphs which lead to the factorization of vacuum graphs, the notion of the generating functional of *connected* graphs and the notion of *effective action* – the generator of *one-particle irreducible* graphs. Since all the graphs can be obtained by simple combinatorics of irreducible graphs, this division is rather useful.

10.1 Factorization of vacuum graphs

First of all, vacuum diagrams are those which do not have external lines. External lines in the diagram describing the multi-point Green's function $\langle 0 \,|\, \mathbb{T}\,\hat\varphi_1...\hat\varphi_n \hat S \,|\, 0 \rangle$ end up on the spacetime points x_k of $\hat\varphi_k = \hat\varphi(x_k)$. It turns out that the vacuum graphs always factor out as one overall factor from every Green's function. This follows from simple combinatorics of contractions between the operators inside the action $S_{\rm int}$ and the operators belonging to the chain $\hat\varphi\hat\varphi_1...\hat\varphi_n$,

$$\begin{aligned}
\langle 0 \,|\, \mathbb{T}\,\hat\varphi\hat\varphi...\hat\varphi \hat S \,|\, 0 \rangle &= \sum_{V=0}^{\infty} \frac{1}{V!} \sum_{\substack{\rm all\\ \rm contractions}} \big(iS_{\rm int}\big)^V \hat\varphi\hat\varphi...\hat\varphi \\
&= \sum_{V=0}^{\infty} \frac{1}{V!} \sum_{M=0}^{V} C_V^M \Bigg[\sum_{\substack{\rm all\\ \rm contractions}} \big(iS_{\rm int}\big)^M \Bigg] \Bigg[\sum_{\substack{\varphi S_{\rm int}\\ \rm contractions}} \big(iS_{\rm int}\big)^{V-M} \hat\varphi\hat\varphi...\hat\varphi \Bigg] \\
&= \Bigg[\sum_{M=0}^{\infty} \sum_{\substack{\rm all\\ \rm contractions}} \frac{\big(iS_{\rm int}\big)^M}{M!} \Bigg] \Bigg[\sum_{V=0}^{\infty} \sum_{\substack{\varphi S_{\rm int}\\ \rm contractions}} \frac{\big(iS_{\rm int}\big)^V}{V!} \hat\varphi\hat\varphi...\hat\varphi \Bigg] \\
&= Z[\,0\,] \,\langle 0 \,|\, \mathbb{T}\,\hat\varphi\hat\varphi...\hat\varphi \hat S \,|\, 0 \rangle_{\,\rm non-vacuum},
\end{aligned} \tag{10.3}$$

where in the last factor $\varphi S_{\rm int}$-contractions by definition contain at least one contraction between φ's and $S_{\rm int}$ and $C_V^M = V!/M!\,(V-M)!$ is the number of possible divisions of the product of V factors into the products of M and $V-M$ factors. Here $Z[\,0\,] = \langle 0 \,|\, \hat S \,|\, 0 \rangle$ is the vacuum to vacuum amplitude given by the set of vacuum diagrams. Thus, non-vacuum Green's functions read

$$\langle 0 \,|\, \mathbb{T}\,\hat\varphi\hat\varphi...\hat\varphi \hat S \,|\, 0 \rangle_{\,\rm non-vacuum} = \frac{1}{Z[\,0\,]}\, \frac{\delta^n Z[\,J\,]}{\delta J_1...\delta J_n}\bigg|_{\,J=0}. \tag{10.4}$$

Division by $Z[\,0\,]$ removes all vacuum diagrams.

Now consider connected diagrams. For a simplest case of the interaction action cubic in the fields

$$S_{\rm int} = \frac{1}{3!}\, S_{abc}\varphi^a\varphi^b\varphi^c, \tag{10.5}$$

we have in the second order of perturbation theory in $S_{\rm int}$

$$\langle \mathbb{T}\,\hat\varphi_1\hat\varphi_2 \rangle \equiv \langle 0 \,|\, \mathbb{T}\,\hat\varphi_1\hat\varphi_2 \hat S \,|\, 0 \rangle = \exp\Big(-\frac{i}{2}\,\frac{\delta}{\delta\varphi} G \frac{\delta}{\delta\varphi}\Big)\,\varphi_1\varphi_2 \,\Big[\, 1 + \frac{i}{3!}\, S^{(3)}\varphi^3 + \frac{1}{2}\,\Big(\frac{i}{3!}\, S^{(3)}\varphi^3\Big)^2 + ...\Big], \tag{10.6}$$

which is easier to draw than to explicitly write. We have it on Fig.6, where all the elements of diagrams are explained including the mean field $\langle \varphi \rangle$, in terms of which the disconnected term is factorized as a product $\langle \varphi_1 \rangle \langle \varphi_2 \rangle$,

The *mean* field is shown on Fig.7 in the lowest order of perturbation theory. It reads as the following contraction of the 3-vertex with two propagators,

$$\langle \hat\varphi^a \rangle \equiv \langle 0 \,|\, \mathbb{T}\,\hat\varphi^a_I\, \hat S \,|\, 0 \rangle = -\frac{i}{2}\, G^{ab} S_{bcd} G^{dc} + ..., \tag{10.7}$$

One propagator forms the external line ending at the entry a of $\langle \hat\varphi^a \rangle$. The second propagator contracts the two indices of the vertex and forms a loop. Of course, everywhere here contractions of condensed indices includes spacetime integration.

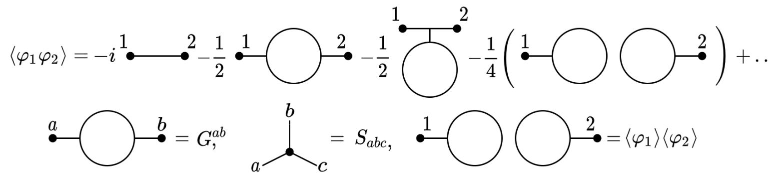

Figure 6: Diagrammatic representation of the two-point Green's function in the second order of perturbation theory. Last term is disconnected.

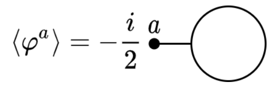

Figure 7: Mean field diagram.

10.2 Connected and one-particle irreducible graphs. Effective action

It turns out that one can get rid of the disconnected piece of $\langle\, \varphi_1 \,\rangle\langle\, \varphi_2 \,\rangle$ in Fig.6 by going over to the generating functional of connected Green's function $Z[\,J\,] \to W[\,J\,]$ which is just the logarithm of $Z[\,J\,]$,

$$W[\,J\,] = \frac{1}{i}\,\ln Z[\,J\,], \quad Z[\,J\,] = e^{i\,W[\,J\,]}. \tag{10.8}$$

For the two-point case this gives

$$W^{12} \equiv G^{12}_{\rm connected} = \frac{\delta^2 W[\,J\,]}{\delta J_1\, \delta J_2} = i\,\big(\,\langle\, \hat\varphi_1 \hat\varphi_2 \,\rangle - \langle\, \hat\varphi_1 \,\rangle\langle\, \hat\varphi_2 \,\rangle\,\big), \tag{10.9}$$

which cancels the disconnected graph in Fig.6. The same can be observed for multi-point connected Green's functions – they are generated by higher order functional derivatives of $W[\,J\,]$.

Problem 10.2. Prove (10.9).

The next step is to consider *one-particle irreducible* Green's functions. Their graphs have the property that cutting down any single of their propagators does not make the total graph disconnected. For example, on Fig.8 the total graph consists of two one particle irreducible diagrams $\mathrm{OPI}_1(x_1, ...x_n, z)$ and $\mathrm{OPI}_2(u, y_1, ...y_m)$ connected by the propagator,

$$\int dz\, du\, \mathrm{OPI}_1(x_1, ...x_n, z)\, G(z, u)\, \mathrm{OPI}_2(u, y_1, ...y_m). \tag{10.10}$$

The knowledge of OPI ingredients of the full diagram of course allows one to calculate it by this rule as a whole. OPI diagrams by their definition do not have external lines-propagators, because their cutting would destroy their irreducible nature. For this reason OPI graphs on Fig.8 have only small black dots corresponding to spacetime entries $x_1, .., y_1, ...$, etc. Different structure of external entries of OPI Green's functions manifest themselves in the different character of their generating functional. It is not a functional of the source J, but rather a functional of the mean field $\phi \equiv \langle\, \hat\varphi \,\rangle$ the simplest version of which has already been introduced above.

Introduce the off-shell version of the mean field as the following functional of the source J

$$\phi^a = \phi^a[\,J\,] = \langle\, 0\,|\, \mathbb{T}\, \hat\varphi^a_I\, \hat S[\,J\,]\,|\, 0\,\rangle = \frac{1}{Z[\,J\,]}\,\frac{\delta Z[\,J\,]}{i\delta J_a}\,\Big|_{\,J\neq 0} = \frac{\delta W[\,J\,]}{\delta J_a}, \tag{10.11}$$

and assume that this relation is invertible, which means that the source can be expressed as a functional of the mean field $J_a = J_a[\,\phi\,]$. Then the generating functional of the OPI Green's functions is the Legendre transform of $W[\,J\,]$

$$\varGamma[\,\phi\,] = \big(\,W[\,J\,] - J_a \phi^a\,\big)\,\Big|_{\,J=J[\,\phi\,]}. \tag{10.12}$$

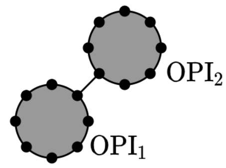

Figure 8: Two one-particle irreducible diagrams connected by a propagator.

This functional is also called *effective action*, though this term is always used in the different sense – as the action in effective field theory, which is valid in the range of energies below a certain cutoff and the result of integrating out high-energy quantum fields.

Effective action generates OPI diagrams or proper (full) quantum vertices

$$\Gamma_{a_1...a_n} = \frac{\delta^n \Gamma[\,\phi\,]}{\delta\phi^{a_1}...\delta\phi^{a_n}}. \tag{10.13}$$

Let us now see that these vertices and the effective action $\Gamma[\,\phi\,]$ itself are represented by OPI diagrams. To begin with, from the properties of the Legendre transform it follows that effective action yields effective equations of motion for the mean field

$$\frac{\delta\Gamma[\,\phi\,]}{\delta\phi^a} = -J_a. \tag{10.14}$$

Problem 10.3. Prove (10.14).

Note that this is the direct analogue of the classical equations of motion in the presence of sources under the replacement of the classical field by the mean field and the replacement of the classical action by the quantum effective action, $\varphi \to \phi$, $S[\,\varphi\,] \to \Gamma[\,\phi\,]$. For $\phi = \phi[\,J\,]$ this is the identity valid for any value of the source $\delta\Gamma[\phi]/\delta\phi\,|_{\,\phi=\phi[J]} = -J$, which can be differentiated with respect to J. Therefore,

$$\Gamma_{ab}\,\frac{\delta\phi^b}{\delta J_c} = -\delta^c_a, \quad \frac{\delta\phi^a}{\delta J_b} = -\big(\Gamma_{ba}\big)^{-1} = \mathcal{G}^{ab}, \tag{10.15}$$

whence

$$\frac{\delta^2 W}{\delta J_a \delta J_b} = -\big(\Gamma_{ba}\big)^{-1} = \mathcal{G}^{ab}, \tag{10.16}$$

where $\mathcal{G}^{ab}$ is the full quantum propagator – the classical Green's function G^{ab} dresses by the full set of quantum corrections. But we know that W generates all connected diagrams so that this second order functional derivative can be rewritten as an infinite sum of diagrams, where the blotted circle denotes the sum of all OPI diagrams

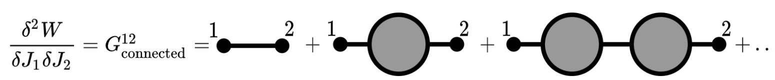

Figure 9: Self energy

with two entries (it cannot contain non-OPI graphs, because the latter are all included in the sum of terms on this Fig.9). If we denote this shaded circle operator as Σ, then the infinite sum of this figure can be resummed as the geometric progression

$$\frac{\delta^2 W}{\delta J_1 \delta J_2} = G + G\Sigma G + G\Sigma G\Sigma G + ... = G\,\frac{1}{1-\Sigma G}. \tag{10.17}$$

On the other hand $G = -(S_{\phi\phi})^{-1}$, so that $\delta^2 W/\delta J_1\delta J_2 = -\big(\,S_{\phi\phi} + \Sigma\,\big)^{-1} = -\big(\,\Gamma_{\phi\phi}\,\big)^{-1}$, where we recalled Eq.(10.16). Therefore

$$\Gamma_{ab} = \frac{\delta^2\Gamma}{\delta\phi^a\delta\phi^b} = S_{ab} + \Sigma_{ab}, \tag{10.18}$$

and $\Sigma[\phi]$ can be interpreted as a quantum part of the full quantum effective action ,

$$\varGamma[\phi] = S[\phi] + \varSigma[\phi], \tag{10.19}$$

and $\varSigma_{ab} = \delta^2\varSigma/\delta\phi^a\delta\phi^b$ is usually called *self-energy operator*.

Problem 10.4. Prove that $\varGamma_{abc} = \varGamma_{ad}\varGamma_{bf}\varGamma_{ce}\,\frac{\delta^3 W}{\delta J_d \delta J_f \delta J_e} = \varGamma_{ad}\varGamma_{bf}\varGamma_{ce}\,G^{dfe}_{\text{connected}}$.

Therefore, the exact connected 3-point Green's function expresses via the full 3-vertex $\varGamma_{\phi\phi\phi}$ and full quantum propagators $\mathcal{G}^{\phi\phi}$,

$$G^{abc}_{\text{connected}} = -\mathcal{G}^{ad}\mathcal{G}^{bf}\mathcal{G}^{ce}\,\varGamma_{dfe}\,. \tag{10.20}$$

Similar expressions hold for higher order connected Green's functions. Diagrammatically they are composed of the full propagators $\mathcal{G}^{\phi\phi}$ joining full vertices $\varGamma_{\phi_1\phi_2...\phi_n}$. In terms of these OPI ingredients connected Green's functions have the form of the trees – in contrast to the Feynman diagrams built of classical (usually called tree level) propagators $G^{\phi\phi}$ and classical (tree level) vertices $S_{\phi_1\phi_2...\phi_n}$ they do not contain loops. Full propagators and full vertices imply infinite resummation of Feynman diagrams in terms of tree level objects – this is a complicated extension of summation via the geometric progression made above.

To summarize what we have got thus far is as follows:

i) non-vacuum diagrams – coefficient functions of S-matrix,

$$G^{a_1...a_n} = \frac{1}{Z[J]}\,\frac{\delta^n Z[J]}{\delta J_{a_1}...\delta J_{a_n}}\,\bigg|_{J=0}\,; \tag{10.21}$$

ii) connected Green's functions and their generating functional

$$W^{a_1...a_n} = \frac{\delta^n W[J]}{\delta J_{a_1}...\delta J_{a_n}}, \quad W = \frac{1}{i}\ln Z; \tag{10.22}$$

iii) OPI vertices and their generating functional – effective action,

$$\varGamma_{a_1...a_n} = \frac{\delta^n \varGamma[\phi]}{\delta\phi^{a_1}...\delta\phi^{a_n}} \quad \varGamma[\phi] = \big(W[J] - J_a\phi^a\big)_{J=J[\phi]}. \tag{10.23}$$

Green's functions (nonvacuum and connected) look very much like contravariant tensors with upper indices on configuration space of the theory[7], whereas the OPI vertex functions have lower "covariant" condensed indices. Diagrammatically this looks as the amputation of external propagator lines from the connected diagrams (along with resummation which makes the diagram irreducible with respect to cutting the propagator lines).

10.3 Semiclassical (loop) expansion and background field formalism

Another type of perturbation theory is semiclassical expansion. This is neither the perturbation expansion in S_{int} (the number of vertices) nor in the number of external lines. There is a natural parameter measuring the deviation of the theory from its classical limit – the Planck constant $\hbar$. Expansion in powers of $\hbar$ is called semiclassical expansion.

To begin with, we reinsert $\hbar$ in the path integral for the generating functional

$$\exp\frac{i}{\hbar}W[J] = \int D\varphi\,\exp\frac{i}{\hbar}\big(S[\varphi] + J_a\varphi^a\big) \tag{10.24}$$

and apply to it the stationary phase method of asymptotic expansion in the limit of $\hbar \to 0$. The essence of the method in the case of finite-dimensional integral

$$\int\limits_{-\infty}^{\infty} dx\,\exp\frac{i}{\hbar}f(x) \tag{10.25}$$

[7]Strictly speaking, they are not tensors because under diffeomorphisms of the configuration space, $\varphi \to \varphi'[\varphi]$, they transform inhomogeneously. Their tensor covariantization can be achieved within the so-called Vilkovisky-DeWitt effective action formalism.

consists in the expansion of the function $f(x)$ in power series in the vicinity of the stationary point x_0, $f'(x_0)=0$,

$$f(x)=f(x_0)+\frac{1}{2}\,f''(x_0)\,(x-x_0)^2+\frac{1}{3!}\,f'''(x_0)\,(x-x_0)^3+...\,, \tag{10.26}$$

and in the subsequent expansion of the integrand in powers of everything but the quadratic part of $f(x)$. The inttegral then reduces to the infinite sum of Gaussian momenta integrals

$$\begin{aligned}\int\limits_{-\infty}^{\infty}dx\,\exp\frac{i}{\hbar}\,f(x)&=e^{if_0/\hbar}\int\limits_{-\infty}^{\infty}d\Delta\,e^{if_0''\Delta^2/\hbar}\,\Big[1+\frac{i}{3!\hbar}\,f_0'''\Delta^3+\frac{i}{4!\hbar}\,f_0''''\Delta^4+\frac{1}{2}\Big(\frac{if_0'''}{3!\hbar}\Big)^2\Delta^6+...\Big]\\&=e^{if_0/\hbar}\Big(\frac{2i\pi\hbar}{f_0''}\Big)^{1/2}\,\Big[1+\frac{i}{3!}\,f_0'''\frac{\langle\Delta^3\rangle}{\hbar}+\frac{i}{4!}\,f_0''''\frac{\langle\Delta^4\rangle}{\hbar}-\frac{1}{2(3!)^2}\,(f_0''')^2\,\frac{\langle\Delta^6\rangle}{\hbar^2}+...\Big],\end{aligned} \tag{10.27}$$

where $f_0=f(x_0)$, $f_0''=d^2f/dx^2|_{x=x_0}$, etc., $\Delta=x-x_0$, and

$$\langle\Delta^n\rangle=\frac{\int d\Delta\,e^{if_0''\Delta^2/\hbar}\,\Delta^n}{\int d\Delta\,e^{if_0''\Delta^2/\hbar}}. \tag{10.28}$$

Odd Gaussian momenta integrals vanish, $\langle\Delta^{2n+1}\rangle=0$, while even ones are expressed as products of pairwise contractions

$$\langle\Delta^{2n}\rangle\sim\underbracket{\Delta\Delta}...\underbracket{\Delta\Delta}\sim\hbar^n,\quad \underbracket{\Delta\Delta}=i\hbar\big(f_0''\big)^{-1}. \tag{10.29}$$

Therefore, in the series (10.27) the powers of $\hbar$ grow in the numerator faster than in the denominator, in particular $\langle\Delta^4\rangle/\hbar\sim\langle\Delta^6\rangle/\hbar^2\sim\hbar$, etc. Therefore this is the expansion in powers of $\hbar$.

Apply this technique to field theory

$$x\to\varphi^a,\quad dx\to D\varphi,\quad f(x)\to S[\,\varphi\,]+J_a\varphi^a,\quad x_0\to\varphi_0,\quad \frac{\delta S[\,\varphi_0\,]}{\delta\varphi_0}=-J, \tag{10.30}$$

$$\underbracket{\Delta\Delta}=i\hbar/f_0''\ \to\ \underbracket{\Delta^a\Delta^b}=-\hbar\,G^{ab}[\,\varphi_0\,]. \tag{10.31}$$

Note that in view of (10.30) $\varphi_0=\varphi_0[\,J\,]$ is the functional of the source as a nontrivial solution of the classical equation of motion. Correspondingly, the chronological contraction $G^{ab}[\,\varphi_0\,]$ is the Green's function of the Hessian of the action on a nontrivial classical *background*,

$$\frac{\delta^2 S[\,\varphi\,]}{\delta\varphi^a\,\delta\varphi^b}\,\Big|_{\varphi_0}\,G^{bc}[\,\varphi_0\,]=-\delta^c_a. \tag{10.32}$$

So the meaning of $\varphi_0[\,J\,]$ here is different from empty flat spacetime background φ_0 in Eq.(8.45).

By using these objects we again arrive at the Feynman diagrams in coordinate space , but with more complicated elements of diagrammatic technique – propagator $G[\,\varphi_0\,]$ and vertices $S_{a_1...a_n}[\,\varphi_0\,]=\delta^n S_{\rm int}/\delta\varphi^{a_1}...\delta\varphi^{a_n}\,|_{\varphi_0}$ defined on the background $\varphi_0[\,J\,]$. The generating functional of connected Green's functions $W[\,J\,]$ becomes given by the set of Feynman graphs *without* external lines or points (just like vacuum diagrams above). The dependence on J enters the answer through $\varphi_0[\,J\,]$,

$$W[\,J\,]=\bar W[\,\varphi_0[\,J\,]\,]. \tag{10.33}$$

The functional $\bar W[\,\varphi_0\,]$ is sometimes called the background field functional.

To the second order in $\hbar$ inclusive the answer for $\bar W[\,\varphi_0\,]$ is graphically shown on Fig.10, where the first two terms represent the tree-level part and the rest is the set of the diagrams with lines and vertices computed on the background field φ_0. The one-loop circle denotes the log of the functional determinant of the Hessian of the action on this background,

$$\ln{\rm Det}\,S_{ab}[\,\varphi_0\,]={\rm Tr}\,\ln S_{ab}[\,\varphi_0\,], \tag{10.34}$$

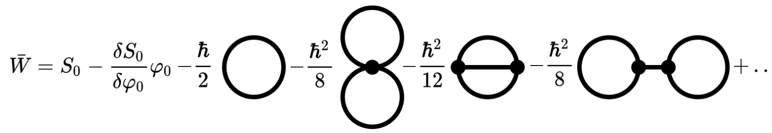

Figure 10: Background field functional in the two-loop approximation.

two-loop eight, nut (or sunset) and the dumbbell one-particle reducible diagrams respectively read as

$$G^{ab}\, S_{abcd}\, G^{cd}, \quad S_{a_1b_1c_1} G^{a_1a_2} G^{b_1b_2} G^{c_1c_2} S_{a_2b_2c_2}, \quad G^{a_1b_1}\, S_{a_1b_1c_1} G^{c_1c_2}\, S_{c_2b_2a_2}\, G^{a_1b_1}. \tag{10.35}$$

Problem 10.5. Derive this representation.

Now, notice that in this diagrammatic technique every vertex carries the factor $\hbar^{-1}$ and every propagator (10.32) carries $\hbar^{+1}$, so that the total power is

$$\hbar^{L-V} = \hbar^{l-1}, \tag{10.36}$$

where L is the number of lines, V is the number of vertices and $l = L - V + 1$ is the number of loops in the Feynman graph. Therefore, semiclassical expansion is the expansion in the number of loops. In fact, this is a resummation of the original perturbation theory – for a given fixed number of loops l we sum up all diagrams with all possible numbers of vertices and all possible numbers of external points (with the background field φ_0 sitting on these external points).

10.4 Effective action in one and two-loop approximations

Let us apply this loop expansion to the effective action $\mathit{\Gamma}[\,\phi\,]$. First, derive in a closed form the equation for it. From the path integral representation (10.24) of $W[\,J\,]$ and the Legendre transform to $\mathit{\Gamma}[\,\phi\,]$ this equation reads

$$\exp\frac{i}{\hbar}\,\mathit{\Gamma}[\,\phi\,] = \int D\varphi\,\exp\frac{i}{\hbar}\Big(S[\,\varphi\,] - \frac{\delta\mathit{\Gamma}[\,\phi\,]}{\delta\phi}\,(\varphi-\phi)\Big) \tag{10.37}$$

Problem 10.6. Derive this equation.

It is important to note that we have two fields in the integrand – the quantum (integration) field φ and the mean field ϕ. Their difference $\varphi - \phi$ is the quantum fluctuation in the vicinity of the mean field.

How to find $\mathit{\Gamma}[\,\phi\,]$ from this equation? An obvious difficulty is that $\delta\mathit{\Gamma}/\delta\phi$ entering the integrand is unknown. But if we are interested in the $\hbar$-expansion, the solution can be found by iterations – by substituting at every step the answer for $\mathit{\Gamma}[\,\phi\,]$ obtained in the previous order in $\hbar$. The starting point is the classical action, $\mathit{\Gamma}[\,\phi\,] = S[\,\phi\,] + O(\,\hbar\,)$ (see Eq.(10.19)). This iteration method is efficient because the quantum fluctuation $\varphi - \phi = O(\,\hbar\,)$, which justifies this substitution for every transition from l-th loop order to the $(l+1)$-st one.

Consider this solution in the one-loop approximation. Substitute $\mathit{\Gamma}_{\rm tree}[\,\phi\,] = S[\,\phi\,]$ into the right hand side of (10.37) and apply stationary phase method. The equation for a stationary point φ_0 takes the simple form $\delta S[\,\varphi_0\,]/\delta\varphi_0 = \delta S[\,\phi\,]/\delta\phi$ and, under the assumption of the non-degeneracy of the Hessian of the classical action $S_{\varphi_0\varphi_0}$, has an obvious solution $\varphi_0 = \phi$. Now expand the exponential in the integrand around this point in powers of the quantum fluctuation $\Delta = \varphi - \phi$ and see that the integral reduces to the Gaussian integral over Δ, because the linear in Δ term cancels out,

$$S[\,\varphi\,] - \frac{\delta S[\,\phi\,]}{\delta\phi}\,(\varphi-\phi) = S[\,\phi\,] + \frac{1}{2}\,S_{\phi\phi}[\,\phi\,]\,\Delta^2 + O(\Delta^3). \tag{10.38}$$

As a result

$$\mathit{\Gamma}[\,\phi\,] = S[\,\phi\,] - \frac{\hbar}{2i}\,{\rm Tr}\ln\frac{\delta^2 S[\,\phi\,]}{\delta\phi\,\delta\phi} + O(\hbar^2), \tag{10.39}$$

and the one-loop contribution to effective action is

$$\mathit{\Gamma}_{\rm one-loop}[\,\phi\,] = -\frac{\hbar}{2i}\,{\rm Tr}\ln\frac{\delta^2 S[\,\phi\,]}{\delta\phi\,\delta\phi}, \tag{10.40}$$

which is graphically represented on Fig.11. Here the loop with a solid line propagator is in the external field with

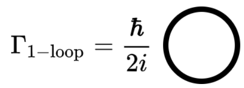

Figure 11: One-loop effective action.

the propagator $G^{\phi\phi}[\,\phi\,]$ – the inverse of $-S_{\phi\phi}[\,\phi\,] \neq -S_{\phi\phi}[\,0\,]$. Of course, it can be expanded in powers of the field in accordance with the expansion of the action Hessian, $S_{\phi\phi}[\,\phi\,] = S_{\varphi\varphi}[\,0\,] + S_{\phi\phi\phi}[\,0\,]\,\phi + (1/2)S_{\phi\phi\phi\phi}[\,0\,]\,\phi^2 + ...$, which leads to the expansion shown on Fig.12. Solid line propagator is a propagator in the external filed ϕ, thin line propagators are at zero background field, black blots denote the factors of the background field attached to the loop via the vertices. Here we use obvious variational relations for ${\rm Tr}\ln S_{\phi\phi}$ and the Green's function

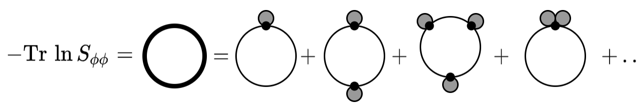

Figure 12: Expansion of the background field loop in powers of the field.

$G[\,\phi\,] = -(S_{\phi\phi}[\,\phi\,])^{-1}$, written down in supercondensed notations (the contraction of omitted condensed indices should be obvious)

$$\frac{\delta}{\delta\phi}\,{\rm Tr}\ln S_{\phi\phi} = -{\rm Tr}\,\big(S_{\phi\phi\phi}G^{\phi\phi}\big),\quad \frac{\delta}{\delta\phi}\,G^{\phi\phi} = G^{\phi\phi}S_{\phi\phi\phi}G^{\phi\phi},\quad \frac{\delta}{\delta\phi}\,S_{\phi\phi\phi} = S_{\phi\phi\phi\phi}. \tag{10.41}$$

These relations teach us that every functional differentiation with respect to ϕ is the insertion of the vertex or attaching extra prong to the already existing vertex. So the solid line loop represents infinite resummation of one loop diagrams with a growing number of ϕ-s attached to this loop.

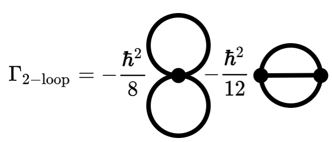

Figure 13: Two-loop part of effective action.

In the second, two-loop, iteration for the effective action one should substitute into the integrand of Eq.(10.37) the one-loop approximation for $\mathit{\Gamma}[\,\phi\,]$ with

$$\frac{\delta\mathit{\Gamma}}{\delta\phi^a}\,\bigg|_{\rm one-loop} = \frac{\delta S}{\delta\phi^a} + \frac{\hbar}{2\,i}\,S_{acd}G^{dc} \tag{10.42}$$

and again perform a Gaussian integration in the vicinity of the stationary point $\varphi = \phi$. The result for the two-loop part of the effective action reads as shown in Fig.13,

$$\mathit{\Gamma}_{2-{\rm loop}} = -\frac{\hbar^2}{8}\,G^{ab}\,S_{abcd}\,G^{cd} - \frac{\hbar^2}{12}\,S_{abc}G^{ad}G^{be}G^{cf}S_{def}. \tag{10.43}$$

These eight and nut (sunset) diagrams are represented by the contractions of propagators and vertices of Eq.(10.35) and coincide with those in the background field functional $\bar W[\,\phi\,]$ of Fig.10. The one particle reducible dumbbell diagram of $\bar W[\,\phi\,]$ gets cancelled in $\mathit{\Gamma}_{\rm two-loop}$ due to the one-loop contribution in (10.42) – the tadpole structure of the mean field (10.7) with amputated external line. This is fully consistent with the fact that $\mathit{\Gamma}[\,\phi\,]$ is a generating functional of OPI diagrams.

Problem 10.7. Derive this expression for two-loop effective action and show cancellation of the dumbbell diagram.

Lecture 11. Quantization of gauge theories

- Canonical Faddeev-Popov path integral
- Gauge independence of the canonical path integral
- The Lagrangian form of the path integral in gauge theories
- The relation between canonical and Lagrangian gauge transformations
- Closure of the gauge algebra and gauge independence of the Lagrangian path integral

Let us now apply the above functional formalism to quantization of gauge theories. As we know from the first five lectures, the physical sector of a gauge invariant theory arises after performing a gauge fixing procedure which consists of imposing unitary gauge conditions and solving the full set of constraints – original first class constraints and imposed gauges. We solve these constraints with respect to original phase space variables (q^i, p_i) and Lagrange multipliers λ^μ in terms of physical degrees of freedom, that is physical phase space variables (ξ^A, π_A). In terms of these variables the action of the theory takes a usual canonical form with some physical Hamiltonian $H(\xi, \pi, t)$ which, depending on the choice of gauge conditions, can be explicitly time dependent and in field theory almost always nonlocal in space,

$$S_{\rm phys}[\,\xi, \pi\,] = \int dt\,\big(\pi_A \xi^A - H(\xi, \pi, t)\big). \tag{11.1}$$

Then the theory can be quantized along standard rules of canonical quantization and canonical commutation relations. Quantum theory can be described by the generating functional for chronological products of physical operators $\hat{\xi}^a(t)$, as it was done before. The canonical path integral representation for this generating functional reads

$$Z[\,J\,] = \int D\xi\, D\pi\, \exp\Big(iS_{\rm phys}[\,\xi, \pi\,] + i\int dt\, J_A(t)\xi^A(t)\Big), \tag{11.2}$$

where the phase space integration runs with boundary conditions discussed above.

In this form, however, the theory is not sufficiently manageable, because it poses many questions which stay without answers. To begin with, in this formalism many original symmetries become hidden, like Lorentz invariance – the corner stone of pioneering progress in quantum electrodynamics – which gets lost in the canonical formalism. The theory is spatially nonlocal, which makes its renormalization properties very complicated and again not manifest. Everything in this reduced phase space method explicitly depends on a particular choice of gauge fixing procedure, and a natural question arises how do quantum effects in different gauges, but in one and the same physical theory, are related to one another. Another problem is how one can reformulate the theory in initial gauge field variables – fields of the Lagrangian action $g^a = (q^i(t), \lambda^\mu(t))$ – without explicitly solving the constraints (which is practically impossible to do exactly).

11.1 Canonical Faddeev-Popov path integral

The reformulation of the above type, which can be a step forward to the resolution of the above questions, is possible via transition in the path integral (11.2) to the original phase space variables and Lagrange multipliers of the canonical formalism of a gauge theory. As we remember from Lecture 4 the Liouville integration measure in the physical phase space,

$$D\xi\, D\pi = \prod_t d\xi(t)\, d\pi(t), \tag{11.3}$$

is the result of reduction of the range of integration over the original phase space of canonical coordinates q^i and conjugated momenta p_i to the subspace of first class constraints $T_\mu(q,p) = 0$ and auxiliary gauge conditions $\chi^\mu(q,p) = 0$. If one formally denotes the dimensionality of phase space by $2n$, $i = 1, 2, ...n$, and the number of first class constraints m, $\mu = 1, 2, ...m$ (m and n are actually infinite and correspond to continuous ranges in field theories), then the $2(n-m)$ dimensional measure of physical phase space, $A = 1, 2, ...n-m$, is the following projection of the original Liouville measure to the Liouville measure on this subspace by delta functions of first class constraints T_μ and gauge conditions χ^μ,

$$\underbrace{d^{n-m}\xi\, d^{n-m}\pi}_{2(n-m)} = \underbrace{d^n q\, d^n p}_{2n}\; \delta(\chi)\, \delta(T)\, J, \tag{11.4}$$

$$\delta(\chi) = \prod_\mu \delta(\chi^\mu), \quad \delta(T) = \prod_\mu \delta(T_\mu), \quad J = \det J^\mu_\nu, \quad J^\mu_\nu = \{\chi^\mu, T_\nu\}. \tag{11.5}$$

Here J^μ_ν is the *canonical Faddeev-Popov operator* – the matrix of Poisson brackets of first class constraints and gauge condition functions – which should be nondegenerate and therefore invertible.

On the other hand, the physical action is a restriction to the constraint subspace of the original canonical action of the theory,

$$S_{\rm phys}[\,\xi, \pi\,] = S[\,q, p, \lambda\,]\,\big|_{T=0,\ \chi=0}\ , \tag{11.6}$$

$$S[\,q, p, \lambda\,] = \int dt\, \big(p_i\, \dot q^i - H_0 - \lambda^\mu T_\mu\big). \tag{11.7}$$

Therefore, the generating functional (11.2) can be rewritten as *canonical Faddeev-Popov path integral*

$$Z[\,J\,] = \int Dq\, Dp\, D\lambda\, \delta[\,\chi\,]\, {\rm Det}\, J\, \exp\Big(iS[\,q, p, \lambda\,] + i\int dt\, J_A(t)\xi^A(t\,|\,q, p)\Big), \tag{11.8}$$

where the *functional* delta function of the first class constraints $\delta[\,T\,]$ is represented as the following integral over the Lagrange multipliers,

$$\delta[\,T\,] = \prod_t \delta\big(T(q(t), p(t))\big) = \int D\lambda\, \exp\Big(-i\int dt\, \lambda^\mu T_\mu\Big). \tag{11.9}$$

The physical coordinates in the source term $\xi^A(t\,|\,q, p)$ are some functions of q and p. Similarly to (11.9) functional delta functions of gauge conditions and functional canonical Faddeev-Popov determinant are defined as products over the moments of time of their ultralocal instantaneous values

$$\delta[\,\chi\,] = \prod_t \delta\big(\chi(q(t), p(t))\big), \tag{11.10}$$

$${\rm Det}\, J = \prod_t \det J^\mu_\nu(q(t), p(t)). \tag{11.11}$$

11.2 Gauge independence of the canonical path integral

Note that the relation (11.4) between the integration measures was derived in Lecture 4 for a particular set coordinate gauge conditions $\chi^\mu = \chi^\mu(q)$, but in fact it holds for a wider class of generic canonical gauges $\chi^\mu(q, p)$ and, moreover, underlies the main property of the Faddeev-Popov path integral – its **on-shell gauge independence**. For the sources switched off one has for small changes of gauge conditions functions,

$$Z_\chi[\,0\,] = Z_{\chi+\delta\chi}[\,0\,], \tag{11.12}$$

To prove this consider infinitesimal change of gauge conditions

$$\chi^\mu \to \chi^\mu + \delta\chi^\mu. \tag{11.13}$$

It can be generated by the canonical transformation of phase space variables with the infinitesimal generating function $\delta\varPhi$ given by a linear combination of first class constraints, $\delta\varPhi = T_\mu \mathcal{F}^\mu$, with some infinitesimal gauge parameters $\mathcal{F}^\mu$

$$\phi = (q, p) \to \phi' = (q', p') = (q + \delta q, p + \delta p), \quad \delta\phi = \{\phi, \delta\varPhi(\phi)\} \tag{11.14}$$

This transformation is ultralocal in time, and it should simulate the change of the gauge conditions functions

$$\chi^\mu(\phi') = \chi^\mu(\phi) + \{\chi^\mu, T_\nu\}\,\mathcal{F}^\nu, \tag{11.15}$$

whence its identification with $\chi^\mu(\phi) + \delta\chi^\mu(\phi)$ gives (remember that $J^\mu_\nu = \{\chi^\mu, T_\nu\}$ is invertible)

$$\mathcal{F}^\nu = J^{-1\nu}_{\;\;\mu}\,\delta\chi^\mu. \tag{11.16}$$

Note that this relation is ultralocal in time, $\delta\chi^\mu(t)$ generates $\mathcal{F}^\mu(t)$ at the same moment in time, because the functional matrix J^μ_ν and its inverse, if treated as spacetime operators are proportional to undifferentiated time delta function, $J^\mu_\nu(t,t') = J^\mu_\nu\,\delta(t-t')$. Therefore, if we assume that the gauge conditions in the definition of S-matrix are not varied at $t_\pm \to \pm\infty$ (what we will do for a time being) then $\mathcal{F}^\nu(t_\pm) = 0$.

Now make the change of integration variable (11.14) in the generating functional (11.8) where we prefer to write the quantum integration measure in terms of the delta function of constraints,

$$D\xi\, D\pi = Dq\, Dp\, \delta[\,\chi\,]\,\delta[\,T\,]\,J_\chi, \tag{11.17}$$

and explicitly indicate by the subscript that the Faddeev-Popov determinant is calculated in the gauge χ^μ. First we just relabel the phase space variables by $\phi' = (q', p')$,

$$Z_\chi[\,0\,] = \int Dq'\,Dp'\,\delta[\,\chi(q',p')\,]\,\delta[\,T(q',p')\,]\,J_\chi[\,q',p'\,]\,\exp i\int dt\Big(p'\,\dot q' - H_0(q',p')\Big), \tag{11.18}$$

and then carefully transform everything to the original (q,p). Let us show that the result will be just a replacement (11.13) of gauge conditions functions.

We have for the exponentiated classical action

$$\int_{t_-}^{t_+} dt\Big(p'\,\dot q' - H_0(q',p')\Big)\,\Big|_{T_\mu=0} = \int_{t_-}^{t_+} dt\Big(p\,\dot q - H_0(q,p)\Big)\,\Big|_{T_\mu=0}, \tag{11.19}$$

because under a canonical transformation the symplectic form transforms by the surface term at $t_\pm$, which vanishes in view of $\mathcal{F}^\nu(t_\pm) = 0$, and $H_0(q',p')\,|_{T=0} = H_0(q,p) + \{H_0, T_\mu\}\mathcal{F}^\mu\,|_{T=0} = H_0(q,p)$ since the Hamiltonian weakly commutes with first class constraints, $\{H_0, T_\mu\} = V^\nu_\mu T_\nu$. Liouville integration measure under canonical transformations is invariant and the delta function of gauge conditions by construction goes over into that of the new ones,

$$Dq'\,Dp' = Dq\,Dp, \quad \delta[\,\chi(q',p')\,] = \delta[\,\chi(q,p) + \delta\chi(q,p)\,]. \tag{11.20}$$

Problem 11.1. Show the above properties of the canonical transformation for symplectic form and Liouville integration measure.

The transformation of the quantum measure is trickier,

$$\delta(T(\phi'))\,J_\chi(\phi') = \delta(T(\phi))\,J_\chi(\phi) + \{\,J_\chi, \delta\varPhi\,\}\,\delta(T) + J_\chi\left[\frac{\partial}{\partial T_\lambda}\delta(T)\right]\{\,T_\lambda, \delta\varPhi\,\}. \tag{11.21}$$

Consider the first two terms and use the variational formula for $\det J^\alpha_\beta$, $\{\,J, \delta\varPhi\,\} = J\,J^{-1\alpha}_{\;\;\beta}\,\{\,J^\beta_\alpha, \delta\varPhi\,\} = J_\chi\,J^{-1\alpha}_{\;\;\beta}\,\{\,\{\chi^\beta, T_\alpha\}, \delta\varPhi\,\}$ and the cyclic Jacobi identity for the double Poisson bracket. The result is

$$\begin{aligned}\delta(T)\,J_\chi + \{\,J, \delta\varPhi\,\}\,\delta(T) &= \delta(T)\,\Big[\,\underbrace{J_\chi + J_\chi\,J^{-1\alpha}_{\;\;\beta}\,\{\,\{\chi^\beta, \delta\varPhi\}, T_\alpha\}}_{J_{\chi+\delta\chi}} - J_\chi\,J^{-1\alpha}_{\;\;\beta}\,\{\{T_\alpha, \delta\varPhi\}, \chi^\beta\}\Big]\\ &= \delta(T)\,\Big[\,J_{\chi+\delta\chi} + J_\chi\,\varOmega^\alpha_\alpha\Big],\end{aligned} \tag{11.22}$$

where we took into account that $\{\chi^\beta, \delta\varPhi\} = \delta\chi^\beta$ and denoted by $\varOmega^\mu_\alpha$ the coefficient of the constraints in the following commutator

$$\{T_\alpha, \delta\varPhi\} = \varOmega^\mu_\alpha T_\mu, \quad \varOmega^\mu_\alpha = U^\mu_{\alpha\lambda}\mathcal{F}^\lambda + \{T_\alpha, \mathcal{F}^\mu\}. \tag{11.23}$$

(remember the first class nature of constraints T_μ weakly commuting with each other, $\{T_\mu, T_\nu\} = U^\alpha_{\mu\nu}\, T_\alpha$).

Problem 11.2. Check Eq.(11.22).

The third term in (11.21) reads

$$J_\chi \Big[\frac{\partial}{\partial T_\alpha}\delta(T)\Big]\,\{\,T_\alpha, \delta\varPhi\,\} = J_\chi \Big[\,\underbrace{T_\mu \frac{\partial}{\partial T_\alpha}\delta(T)}_{-\delta^\alpha_\mu\,\delta(T)}\,\Big]\,\varOmega^\mu_\alpha = -J_\chi\,\varOmega^\alpha_\alpha\,\delta(T) \tag{11.24}$$

and cancels the second term of (11.22), so that we finally have for the quantum measure on the phase space of $\phi = (q,p)$ the transformation

$$\delta(T(\phi'))\,J_\chi(\phi') = \delta(T(\phi))\,J_{\chi+\delta\chi}(\phi). \tag{11.25}$$

Altogether Eq.(11.18) becomes

$$Z_\chi[\,0\,] = \int Dq\,Dp\,D\lambda\,\delta[\,\chi+\delta\chi\,]\,\mathrm{Det}\,J_{\chi+\delta\chi}\,e^{iS[\,q,p,\lambda\,]} = Z_{\chi+\delta\chi}[\,0\,]. \tag{11.26}$$

On-shell Faddeev-Popov path integral is gauge independent at least in the class of gauge conditions functions related by sequence of "small" gauge transformations with nondegenerate canonical Faddeev-Popov operators.

This theorem can be easily extended to off-shell generating functionals if the sources J^I are turned on for some gauge-invariant observables $\mathcal{O}_I$ whose canonical gauge transformation weakly vanishes on the constraint surface in phase space, that is

$$\{\mathcal{O}_I, T_\mu\} = V^{\,\nu}_{I\,\mu}\,T_\nu. \tag{11.27}$$

Then the proof of gauge independence of such generating functional of correlators $\langle\,\mathcal{O}_{I_1}...\mathcal{O}_{I_n}\rangle$,

$$Z[\,J\,] = \int Dq\,Dp\,D\lambda\,\delta[\,\chi\,]\,\mathrm{Det}\,J_\chi\,e^{\,i\,S[\,q,p,\lambda\,]+i\int dt\,J^I(t)\mathcal{O}_I(t)}, \tag{11.28}$$

obviously goes without difficulty in view of gauge invariance of the source term on the subspace of constraints in phase space of the theory, $\delta^{\mathcal{F}}\mathcal{O}_I\,|_{\,T=0} = 0$. Below we will see that weakly vanishing sources in the path integral representation of S-matrix actually represent the example of such gauge-invariant observables.

11.3 The Lagrangian form of the path integral in gauge theories

Transition to the Lagrangian form of the path integral is achieved, as in the non-gauge theory case, by integrating out the canonical momenta. When the Hamiltonian in the canonical action is quadratic in momenta, this can be done exactly by taking the Gaussian integral. For the class of coordinate gauge conditions $\chi^\mu(q)$ the result reads

$$\int Dp\,e^{\,i\,S[\,q,p,\lambda\,]} = \mathrm{const}\,\big[\,\mathrm{Det}\,G^{ij}\,\big]^{-1/2}\,e^{\,i\,S[\,g\,]}, \tag{11.29}$$

where $S[\,g\,] = S[\,q,\lambda\,]$ is the *Lagrangian action* which follows from the canonical action by the substitution of the momentum $p = p_0(q,\dot q,\lambda)$ as a function of coordinates q, their time derivatives and Lagrange multipliers λ

$$S[\,g\,] \equiv S[\,q,\lambda\,] = \int dt\,L(q,\dot q,\lambda) = S[\,q, p_0(q,\dot q,\lambda),\lambda\,], \tag{11.30}$$

$$G^{ij} = \frac{\partial^2}{\partial p_i\,\partial p_j}\big(\,H_0 + \lambda^\mu\,T_\mu\,\big) = \big(\,G_{ij}\big)^{-1}, \quad G_{ij}\,\big|_{\,p=p_0} = \frac{\partial^2 L}{\partial\dot q^i\,\partial\dot q^j}. \tag{11.31}$$

The equation for $p_0(q,\dot q,\lambda)$ is[8]

$$\frac{\delta S[q,p,\lambda]}{\delta p_i(t)}\,\Big|_{\,p=p_0}=\dot q^i-\frac{\partial}{\partial p_i}\,\big(\,H_0+\lambda^\mu\,T_\mu\,\big)\,\Big|_{\,p=p_0}=0. \tag{11.32}$$

Thus, the generating functional takes the Lagrangian form

$$Z[\,J\,]=\int Dg\,\mu[\,g\,]\,\delta[\,\chi(g)\,]\,\mathrm{Det}\,Q[\,g\,]\,e^{\,i\,S[\,g\,]+iJ_a g^a}, \tag{11.33}$$

$$Dg=Dq\,D\lambda,\quad \mu[\,g\,]=\prod_t\big(\,\det G_{ij}\big)^{1/2}=\big(\mathrm{Det}\,G_{ij}\big)^{1/2}, \tag{11.34}$$

$$Q[\,g\,]\equiv Q^\mu_\nu[\,g\,]=J^\mu_\nu\,\big|_{\,p=p_0}\,, \tag{11.35}$$

where $\mu[\,g\,]$ is the contribution of the local measure and $Q^\mu_\nu[\,g\,]$ is the Lagrangian form of the Faddeev-Popov operator. In contrast to the original canonical path integral, the sources are included here to the full set of gauge fields $g^a=(q^i(t),\lambda^\mu(t))$ including the Lagrange multipliers in order to have the possibility of generating the correlation functions of all gauge fields g^a,

$$J_a\,g^a=\int dt\,\big(J_i(t)\,q^i(t)+J_\mu(t)\,\lambda^\mu(t)\big). \tag{11.36}$$

We remind that we use both the spacetime condensed notations (as in g^a) and canonical condensed notations in which indices contractions involve spacetime and space integrations respectively.

Let us briefly discuss the contributions of the local measure $\mu[\,g\,]$ and the Faddeev-Popov determinant $\mathrm{Det}\,Q^\mu_\nu[\,g\,]$. As it was discussed before, the matrix of local measure G_{ij} is ultralocal in time, so that in spacetime condensed notations

$$G_{ij}=G_{ij}(t)\,\delta(t-t'),\quad i\mapsto(i,t),\quad j\mapsto(j,t'), \tag{11.37}$$

$$\mu[\,g\,]=\big(\mathrm{Det}\,G_{ij}\big)^{1/2}=\Big(\prod_t\det G_{ij}(t)\Big)^{1/2}=\exp\Big(\frac12\delta(0)\int dt\,\ln\det G_{ij}(t)\Big). \tag{11.38}$$

The local measure contribution is always divergent and serves to cancel the strongest volume divergences of Feynman graphs. In the dimensional regularization these divergences and the measure contribution altogether vanish and, therefore, physically are unimportant (modulo the subtleties of the regularization mechanism which might lead to quantum anomalies which we will not discuss here).

11.4 The relation between canonical and Lagrangian gauge transformations

To interpret the Faddeev-Popov operator $Q^\mu_\nu[\,g\,]$ note that for coordinate gauge conditions $\chi^\mu(q)$ it describes their canonical gauge transformation generated by the first class constraints (remember the material of Lectures 4 and 5)

$$Q^\mu_\nu[\,g\,]\,\mathcal F^\nu=\{\chi^\mu,T_\nu\}\,\big|_{\,p=p_0}\,\mathcal F^\nu=\delta^{\mathcal F}\chi^\mu\,\big|_{\,p=p_0}. \tag{11.39}$$

On the other hand, there is a map between these Hamiltonian formalism gauge transformations $\delta^{\mathcal F}g^a$ and their Lagrangian counterparts $\Delta^f g^a$, $\delta^{\mathcal F}g^a=\Delta^f g^a$ under a certain relation between the Hamiltonian gauge parameters $\mathcal F^\mu$ and the Lagrangian ones f^α. The structure of $\delta^{\mathcal F}g^a$ is different for q^i and λ^μ – for coordinates

[8] The restriction to coordinate gauge conditions $\chi^\mu(q)$ allows one to avoid the contribution of $\delta[\,\chi(q)\,]$ to the equation for p_0, but for theories with constraints nonlinear in momenta extra dependence on momenta enters through the Faddeev-Popov determinant $\mathrm{Det}\,J^\mu_\nu[\,q,p\,]$ and makes the integral non-Gaussian. Still, for theories with at most one (per spatial point) constraint T_μ quadratic in momenta (the case of Einstein gravity with the Hamiltonian constraint $H_\perp$ among $T_\mu\equiv H_\mu$, see Lecture 4) the integral over p_i can be exactly taken and reduces to the substitution of p_0 in the action. This was shown in *E.S.Fradkin and G.A.Vilkovisky, Quantization of Relativistic Systems with Constraints: Equivalence of Canonical and Covariant Formalisms in Quantum Theory of Gravitational Field, preprint CERN-TH-2332, 1977.* For generic local theories the deviation from this rule reduces to extra perturbative contributions to the *local* measure $\sim\delta(0)$ discussed below.

q^i they are just canonical ones, that is generated by the constraints T_ν and ultralocal in time, whereas for λ^μ they involve time derivatives of $\mathcal{F}^\mu$

$$\delta^{\mathcal{F}} q^i\,\big|_{\,p=p_0} = \frac{\partial T_\mu}{\partial p_i}\bigg|_{\,p=p_0} \mathcal{F}^\mu, \tag{11.40}$$

$$\delta^{\mathcal{F}} \lambda^\mu = \dot{\mathcal{F}}^\mu - U^\mu_{\sigma\nu}\,\lambda^\sigma \mathcal{F}^\nu - V^\mu_\nu\,\mathcal{F}^\nu. \tag{11.41}$$

In the Lagrangian formalism they both can be represented in terms of *gauge generators* R^a_μ as spacetime differential operators with the entries a and μ – spacetime condensed indices. In both spacetime condensed and canonical condensed (with explicit integration over time) this reads as

$$\Delta^f g^a = R^a_\mu\, f^\mu \equiv \int dt'\, R^a_\mu(t,t')\, f^\mu(t'), \tag{11.42}$$

$$R^i_\mu \equiv R^i_\mu(t,t') = \frac{\partial T_\mu}{\partial p_i}\bigg|_{\,p=p_0} \delta(t-t'), \quad i\mapsto(i,t), \quad \mu\mapsto(\mu,t') \tag{11.43}$$

$$R^\sigma_\mu \equiv R^\sigma_\mu(t,t') = \Big(\delta^\sigma_\mu\,\frac{d}{dt} - U^\sigma_{\rho\mu}\,\lambda^\rho - V^\sigma_\mu\Big)\,\delta(t-t'), \quad \sigma\mapsto(\sigma,t), \quad \mu\mapsto(\mu,t'). \tag{11.44}$$

We know how $\mathcal{F}^\mu$ is related to f^α in concrete examples of relativistic particle, Yang-Mills theory and Einstein gravity, and generally $\mathcal{F}^\mu \neq f^\mu$. In what follows we will assume for simplicity that they coincide, which in principle can be attained by a reparametrization of the group manifold, or if necessary in relevant equations additional factors of the matrix $\partial f^\alpha(\mathcal{F})/\partial\mathcal{F}^\mu$ will appear.[9] Then the Faddeev-Popov matrix in (11.39) can be written down as Lagrangian gauge transformation of the gauge conditions functions

$$Q^\mu_\nu[\,g\,] = \frac{\delta\chi^\mu}{\delta g^a}\,R^a_\nu. \tag{11.45}$$

For *coordinate gauges* $\chi^\mu(q)$ this is an ultralocal in time operator $Q^\mu_\nu = J^\mu_\nu(q,p)\,|_{\,p=p_0}\delta(t-t')$. We now want to extend the notion of the Lagrangian path integral to a wider class of gauges and claim that it will be gauge independent (on shell or for gauge-invariant source terms) and, therefore, coincide with the path integral in unitary gauges.

11.5 Closure of the gauge algebra and gauge independence of the Lagrangian path integral

So we define the Faddeev-Popov path integral (11.33) in a generic gauge $\chi^\mu(g) = \chi^\mu(q,\dot{g})$ – local functions of gauge fields g^a and their spacetime derivatives – with the nontrivial quantum measure $\delta[\,\chi(g)\,]\,\mathrm{Det}\,Q^\mu_\nu[\,g\,]$, where the Faddeev-Popov operator defined by Eq.(11.45) is no longer ultralocal in time. Successful choice of gauge conditions will manifestly restore such symmetries as Lorentzian symmetry of the formalism and will guarantee unitarity because the starting point was canonical quantization in the physical sector with a Hermitian Hamiltonian. This is the logic of recovering manifest covariance simultaneously retaining unitarity of the theory.

We will see now that the critical point of this strategy – gauge independence of the path integral – will rely on locality of the theory and the *closure of the algebra of gauge transformations* in Lagrangian formalism. This means that it forms the representation of the group algebra closed with respect to the Lie bracket of gauge generators R^a_μ on the configuration space of gauge fields g^a. In spacetime condensed notations this property reads as

$$R^b_\mu\,\frac{\delta R^a_\nu}{\delta g^b} - R^b_\nu\,\frac{\delta R^a_\mu}{\delta g^b} = C^\lambda_{\mu\nu}\,R^a_\lambda, \tag{11.46}$$

where $C^\lambda_{\mu\nu}$ are the structure constants of the gauge algebra.

The proof of gauge independence of (11.33) proceeds along the lines very similar to those in the canonical formalism. We imitate the change of gauge conditions $\chi^\mu(g)\to\chi^\mu(g)+\delta\chi^\mu(q)$ by the gauge transformation

$$g'^a = g^a + R^a_\mu\,f^\mu, \quad \delta\chi^\mu = \frac{\delta\chi^\mu}{\delta g^a}\,R^a_\nu\,f^\nu = Q^\mu_\nu\,f^\nu \;\;\Rightarrow\;\; f^\mu = Q^{-1\mu}_{\;\;\;\nu}\,\delta\chi^\nu, \tag{11.47}$$

[9] For example, $J^\mu_\nu = Q^\mu_\alpha\,\delta f^\alpha/\delta\mathcal{F}^\mu$, or $\mathrm{Det}J^\mu_\nu = \mathrm{Det}Q^\mu_\alpha \times \mathrm{Det}\big(\delta f^\alpha/\delta\mathcal{F}^\mu\big)$ with the last factor contributing to the local measure.

where $Q^{-1\mu}_{\ \ \nu}$ is the inverse of Q^μ_ν – the Green's function which is supposed to exist because Q^μ_ν is of course non-degenerate.[10] Under this transformation we have

$$S[\,g'\,] = S[\,g\,], \quad \delta[\,\chi(g')\,] = \delta[\,\chi(g) = \delta\chi(g)\,], \quad Dg' = Dg\,\left|\,\frac{Dg'}{Dg}\,\right|, \tag{11.48}$$

$$\begin{aligned} \left|\,\frac{Dg'}{Dg}\,\right| &= \mathrm{Det}\,\frac{\delta g'^a}{\delta g^b} = 1 + \left(R^a_\mu\,Q^{-1\mu}_{\ \ \nu}\,\delta\chi^\nu\right)_{,a} \\ &= 1 + R^a_{\mu,a}\,f^\mu + R^a_\mu\,Q^{-1\mu}_{\ \ \nu}\,\delta\chi^\nu_{,a} - R^a_\mu\,Q^{-1\mu}_{\ \ \alpha}\left(\chi^\alpha_{,ba}\,R^b_\beta + \underline{\chi^\alpha_{,b}\,R^b_{\beta,a}}\right) f^\beta, \end{aligned} \tag{11.49}$$

where comma denotes the functional derivative with respect to the gauge field, $\delta\chi^\mu_{,a} = \delta\chi^\mu/\delta g^a$, $\chi^\alpha_{,ba} = \delta^2\chi^\alpha/\delta g^a\delta g^b$, etc., and we used the formula of variation of the determinant. Similarly

$$Q_\chi[\,g'\,] = Q_\chi[\,g\,]\left[1 + Q^{-1\mu}_{\ \ \alpha}\,R^b_\mu\,\delta\chi^\alpha_{,ba}\,R^b_\beta\,f^\beta + \underline{Q^{-1\mu}_{\ \ \alpha}\,R^b_{\mu,a}\,\chi^\alpha_{,b}\,R^a_\beta\,f^\beta}\right]. \tag{11.50}$$

Problem 11.3. Check Eqs.(11.49)-(11.50)

Combining together Eqs.(11.49)-(11.50) one finds that the terms with $\chi^\alpha_{,ba} = \chi^\alpha_{,ab}$ cancel out, whereas the underlined terms form the Lie bracket of gauge generators

$$\begin{aligned} Q_\chi[\,g'\,]\left|\,\frac{Dg'}{Dg}\,\right| &= Q_\chi\left[1 + Q^{-1\mu}_{\ \ \nu}\,R^a_\mu\,\delta\chi^\nu_{,a}\right] \\ &\quad + Q_\chi\left[R^a_{\mu,a}\,f^\mu + \underline{Q^{-1\mu}_{\ \ \alpha}\,\chi^\alpha_{,b}\left(R^a_\beta\,R^b_{\mu,a} - R^a_\mu\,R^b_{\beta,a}\right) f^\beta}\right] \\ &= Q_{\chi+\delta\chi} + Q_\chi\left[\frac{\delta R^a_\mu}{\delta g^a} + C^\nu_{\mu\nu}\right] f^\mu. \end{aligned} \tag{11.51}$$

Finally, the transformation of the local measure reads

$$\mu[\,g'\,] = \mu[\,g\,]\left[1 + f^\mu\,R^a_\mu\,\frac{\ln\mu}{\delta g^a}\right]. \tag{11.52}$$

Therefore, if we assume that the local measure transforms like

$$R^a_\mu\,\frac{\ln\mu}{\delta g^a} = -\left[\frac{\delta R^a_\mu}{\delta g^a} + C^\nu_{\mu\nu}\right], \tag{11.53}$$

then the total quantum measure has under the transformation (11.47) simulating the change of the gauge the needed form

$$\mu[\,g'\,]\,\delta[\,\chi(g')\,]\,\mathrm{Det}\,Q_\chi[\,g'\,]\,\left|\,\frac{Dg'}{Dg}\,\right| = \mu[\,g\,]\,\delta[\,\chi(g) + \delta\chi(g)\,]\,\mathrm{Det}\,Q_{\chi+\delta\chi}[\,g\,]. \tag{11.54}$$

This finally proves that, similarly to (11.26) the on-shell path integral is gauge independent in the class of gauge conditions functions related by "small" gauge transformations with nondegenerate Faddeev-Popov operators,

$$Z_\chi[\,0\,] = Z_{\chi+\delta\chi}[\,0\,]. \tag{11.55}$$

Note that the gauge transformation of the measure (11.53) is consistent in the sense that both right and left hand sides in field theories are $\delta(0)$-type divergences, which vanish in the dimensional regularization and can be discarded from the very beginning because they do not lead to visible physical effects. Indeed, for diffeomorphism group structure constants

$$C^{\lambda,y}_{\mu,x\ \ \nu,x'} = \delta^\lambda_\mu\delta(x,y)\partial_\nu\delta(x,x') - (\mu,x \leftrightarrow \nu,x'), \quad C^\nu_{\mu\nu} = \partial_\mu\delta(x,x')\,|_{\,x'=x}.$$

[10] Below we will see that in a wide class of gauges Q^μ_ν is a second order differential operator, and its Green's function is consistently defined by the same boundary conditions as those of physical degrees of freedom – positive/negative frequency boundary conditions at future/past infinities.

Similarly $R^a_{\mu,a} \propto \partial\delta(0)$ and $\ln\mu[\,g\,] \propto \delta(0)$ in view of (11.38). The analogue of this measure in the finite-dimensional context (when the ranges of $i = 1,...n$ and $\mu = 1,...m$ are really finite) is the Haar measure of integration over the m-dimensional group manifold. Usually the Lagrangian (or covariant) version of the Faddeev-Popov integral is introduced via the procedure of integration over the group and factorization of the group volume factor. This is rigorous only in the finite-dimensional case and for compact groups with a finite group volume. To avoid both of these difficulties we chose another procedure – derivation of the covariant Faddeev-Popov integral from its Hamiltonian version, which in its turn originates from manifestly unitary quantization in the physical sector.

Lecture 12. Relativistic gauge conditions and gauge ghost fields

- On-shell source term
- t'Hooft trick and gauge breaking term
- Feynman-DeWitt-Faddeev-Popov ghosts
- Boundary conditions for gauge and ghost fields
- Recovery of classical theory, initial conditions and counting physical degrees of freedom

Here we continue the discussion of gauge independence of the Faddeev-Popov path integral and S-matrix, consider various types of gauge-fixing procedure and reinterpret the quantum path integral measure in terms of ghost fields.

12.1 On-shell source term

As in the case of unitary gauges, the gauge independence property (11.55) can be extended off shell if the sources are turned on for gauge invariant observables, $J_a g^a \to J_I \mathcal{O}^I[\,g\,]$, satisfying $R^a_\mu\,\delta\mathcal{O}^I/\delta g^a = 0$. A particular case of such gauge invariant is the on-shell weakly vanishing source term which arises in the construction of S-matrix via the generating functional (8.47), that is $Z[\,J\,]$ with J given by (8.38). To generate S-matrix in the physical sector, the source term of (11.2) should read (see Eq.(8.39))

$$-\int\limits_{-\infty}^{+\infty} dt\,\xi_I(t)\,\overrightarrow{S}_{\xi\xi}\left(\frac{d}{dt}\right)\xi(t) = \Big[\,\xi_I\,\pi - \pi_I\,\xi\,\Big]^{t=+\infty}_{t=-\infty}, \tag{12.1}$$

where $\pi = \pi^0(\xi(t),\dot\xi(t))$ and $\pi_I = \pi^0(\xi_I(t),\dot\xi_I(t))$ are the Lagrangian values of canonical momenta for the quantum (integration) physical field $\xi(t)$ and the interaction picture field $\xi_I(t)$ repectively. We remind that the latter is a solution of linearized equation $S_{\xi\xi}(d/dt)\xi_I(t) = 0$ with the Hessian of the physical action $S_{\xi\xi} \equiv \delta^2 S^{\rm phys}/\delta\xi\delta\xi$, so that after integration by parts only the surface term at $t = \pm\infty$ survives and equals the symplectic form built in terms of the two fields ξ and ξ_I. Natural question arises whether this source term can be transformed to original Lagrangian variables $g^a = (q^i,\lambda^\mu)$ similarly to the transformation $(\xi,\pi) \to (q,p)$ in (11.4)-(11.5) and whether it will be gauge invariant in terms of g^a.

It turns out that this is indeed possible because the source term (12.1) is at infinity $t \to \pm\infty$ where the integration field $\xi(t)$ satisfies positive/negative frequency boundary conditions and can be treated in the linearized approximation. Therefore the reduction to physical sector becomes a linear problem which can be universally solved. Let is show that (12.1) can be rewritten in terms of original Lagrangian fields $g^a = (q^i,\lambda^\mu)$ as

$$-\xi_I\,\overrightarrow{S}_{\xi\xi}\,\xi = -g_I\,\overrightarrow{S}_{gg}\,g = -\int\limits_{-\infty}^{+\infty} dt\,g^a_I(t)\,\overrightarrow{S}_{ab}\left(\frac{d}{dt}\right)g^b(t), \tag{12.2}$$

where $g^a_I = (q^i_I(t),\lambda^\mu_I(t))$ is the set of interaction picture fields – generic solutions of the linearized equations of motion analogous to (8.46) composed of positive/negative frequency basis functions $u^a_A(t) \sim e^{-i\omega_A t}$ and

$u_A^{a*}(t) \sim e^{i\omega_A t}$, $\omega_A > 0$, $|\,t\,| \to \infty$,

$$S_{ab} g_I^b = 0, \quad \hat{g}_I^a = u_A^a \alpha_A + u_A^{a*} \alpha_A^*, \quad u_A^a \sim e^{-i\omega_A t}, \quad \omega_A > 0. \tag{12.3}$$

Here the operator S_{ab} is the Hessian of the classical action on the empty and flat space background g_0,

$$S_{ab} = \frac{\delta^2 S[\,g\,]}{\delta g^a \, \delta g^b}\bigg|_{g_0}, \tag{12.4}$$

and (α_A^*, α_A) are the c-number arguments of the S-matrix normal symbol, which get replaced by creation/annihilation operators in S-matrix and scattering amplitudes (cf. Eqs.(8.51)-(8.52)).

From the form of the action (11.30) with the Lagrangian $L(q, \dot{q}, \lambda)$ it is obvious that in the block structure of this Hessian,

$$S_{ab} = \begin{bmatrix} S_{ij} & S_{i\nu} \\ S_{\mu j} & S_{\mu\nu} \end{bmatrix}, \tag{12.5}$$

the block $S_{\mu\nu} = (\partial^2 L/\partial\lambda^\mu \partial\lambda^\nu)\,\delta(t-t')$ is an ultralocal in time operator, whereas S_{ij} is a second order operator in time derivatives and off-diagonal blocks are the first order operators. Therefore in the Wronskian relation of footnote 4, obtained by integrating over time by parts, $S_{\mu\nu}$-block does not contribute at all and the rest takes the form

$$-g_I^a \overrightarrow{S}_{ab}\, g^b = -\int\limits_{-\infty}^{+\infty} dt \left[g_I^a(t) \overrightarrow{S}_{ab}\, g^b(t) - g_I^a(t) \overleftarrow{S}_{ab}\, g^b(t) \right] = \Big[\, q_I^i \big(\overrightarrow{W} g\big)_i - q^i \big(\overrightarrow{W} g_I\big)_i \Big]_{-\infty}^{+\infty}, \tag{12.6}$$

where

$$\big(\overrightarrow{W} g\big)_i = \overrightarrow{W}_{ib}\, g^b = \delta_g \left(\frac{\partial L}{\partial \dot{q}^i} \right) \equiv \delta_g\, p_i^0(q, \dot{q}, \lambda) \tag{12.7}$$

is the linearized (in the variable g) canonical momentum as a function of g and $\dot{g}$ (and similarly for g_I and $\dot{g}_I$).[11]

Problem 12.1. Check (12.6) and prove that

$$\overrightarrow{W}_{ib}\, g^b = \Big(a_{ij} \frac{d}{dt} + \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} \Big)\, g^j + \frac{\partial^2 L}{\partial \dot{q}^i \partial \lambda^\mu}\, g^\mu, \quad a_{ij} \equiv \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}.$$

Therefore, (12.6) becomes a symplectic form built at $t \to \pm\infty$ in terms of the linearised quantum fields (q^i, p_i) and linear interaction picture fields q_I^i and $p_i^I = (\vec{W} g_I)_i$

$$-g_I^a \overrightarrow{S}_{ab}\, g^b = \Big[\, q_I^i\, p_i - q^i\, p_i^I \,\Big]_{-\infty}^{+\infty}. \tag{12.8}$$

Here we disregard the linearization notation for variables (q^i, p_i) because they are anyway linearized due to boundary conditions on integration field g.

Now, we substitute this source term in the integrand of the canonical or Lagrangian path integral (11.33),

$$Z[\,-g_I^a\, \overrightarrow{S}_{ab}\,] = \int Dg\, \mu[\,g\,]\, \delta[\,\chi(g)\,]\, \mathrm{Det}\, Q[\,g\,]\, e^{\,i\,S[\,g\,] - i\, g_I^a \overrightarrow{S}_{ab} g^b}. \tag{12.9}$$

Then in the coordinate gauge $\chi(g) = \chi(q)$, if we recollect the reduction to physical sector in Lecture 5, we have for linearized fields $q^i = e_A^i \xi^A$ and $\pi_A = e_A^i\, p_i$ (and similar relations for interaction picture variables (ξ_I, π_I)). Here $q^i = e^i(\xi)$ are the embedding functions of the physical configuration space into the space of coordinates

[11] In other words, the interpretation of Eq.(12.6) is that the canonical momentum conjugated to λ^μ is identically vanishing and $\big(\overrightarrow{W} g\big)_\mu = 0$.

q^i, and $e^i_A = \partial e^i/\partial\xi^A$. Therefore the symplectic form (12.8) exactly coincides with the the physical symplectic form in Eq.(12.1),

$$-g^a_I\,\overrightarrow{S}_{ab}\,g^b = \Big[\,\xi^A_I e^i_A\,p_i - \xi^A e^i_A\,p^I_i\,\Big]^{+\infty}_{-\infty} = \Big[\,\xi^A_I\,\pi_A - \xi^A\,\pi^I_A\,\Big]^{+\infty}_{-\infty}. \qquad (12.10)$$

Thus the Faddeev-Popov path integral *in the canonical coordinate gauge* exactly recovers the unitary S-matrix construction in the physical sector.

In order to go beyond this class of gauges we have to perform the procedure of changing the gauge conditions by means of gauge transformations under the path integral sign. Gauge independence will hold if the symplectic source term (12.8) is gauge invariant. To prove this consider

$$-\Delta^{\mathcal{F}}\big(\,g^a_I\,\overrightarrow{S}_{ab}\,g^b\,\big) = \Big[\,q^i_I\,\big(\Delta^{\mathcal{F}}p_i\big) - \big(\Delta^{\mathcal{F}}q^i\big)\,p^I_i\,\Big]^{+\infty}_{-\infty} \qquad (12.11)$$

(note that only quantum fields are transformed). We have

$$\Delta^{\mathcal{F}}q^i = \frac{\partial T_\mu}{\partial p_i}\,\mathcal{F}^\mu, \quad \Delta^{\mathcal{F}}p_i = -\frac{\partial T_\mu}{\partial q^i}\,\mathcal{F}^\mu, \qquad (12.12)$$

whence it follows that

$$\Delta^{\mathcal{F}}\big(\,g^a_I\,\overrightarrow{S}_{ab}\,g^b\,\big) = \Big[\,\frac{\partial T_\mu}{\partial q^i}\,q^i_I + \frac{\partial T_\mu}{\partial p_i}\,p^I_i\,\Big]^{+\infty}_{-\infty} = T^{(1)}_\mu(q_I,p_I)\,\Big|^{+\infty}_{-\infty} = 0, \qquad (12.13)$$

where $T^{(1)}_\mu(q_I,p_I) = -S_{\mu a}g^a_I = 0$ is the linearized constraint which is just a μ-component of equations of motion for interaction picture fields. Thus this weakly vanishing source term is gauge invariant and, therefore, admits the transition of the Faddeev-Popov path integral to other gauges without changing S-matrix.

Problem 12.2. There is a subtlety in using the second of equations (12.12). Point is that p here is actually the Lagrangian expressions for momentumt $p = p_0(q,\dot q,\lambda)$. Therefore, $\Delta^{\mathcal{F}}p^0_i$ should be understood as the result of gauge transformation of all of its arguments q, $\dot q$ and λ by the first class constraints (and special transformation of Lagrange multipliers (11.44)). Show, however, that $\Delta^{\mathcal{F}}p^0_i(q,\dot q,\lambda)$ differs from (12.12), $\Delta^{\mathcal{F}}p^0_i(q,\dot q,\lambda) \neq \delta^{\mathcal{F}}p_i\,|_{p=p_0}$, by the terms proportional to *dynamical* equation of motion for q,

$$\Delta^{\mathcal{F}}p^0_i(q,\dot q,\lambda) = -\frac{\partial T_\mu}{\partial q^i}\,\mathcal{F}^\mu - G_{ij}\,\frac{\partial^2 T_\mu}{\partial p_j\partial p_k}\,\frac{\delta S}{\delta q^k}\,\mathcal{F}^\mu,$$

$$\frac{\delta S}{\delta q^k} = -\dot p_k - \frac{\partial H_0}{\partial q^k} - \lambda^\nu\,\frac{\partial T_\nu}{\partial q^k}, \quad G_{ij} = \Big(\frac{\partial^2 H_0}{\partial p_i\partial p_j} - \lambda^\nu\,\frac{\partial^2 T_\nu}{\partial p_i\partial p_j}\Big)^{-1}.$$

Hint. From the equation $\delta S/\delta p = 0$ find the derivatives of $p^0(q,\dot q,\lambda)$ with respect to all of its arguments. Use them to derive $\Delta^{\mathcal{F}}p_0 = \frac{\partial p_0}{\partial q}\Delta^{\mathcal{F}}q + \ldots$. Employ the commutator relations for constraints, $\{T_\mu,T_\nu\} = U^\sigma_{\mu\nu}T_\sigma$, $\{H_0,T_\nu\} = V^\sigma_\nu T_\sigma$, and their derivatives with respect to canonical momenta. Find the modification of the above relations for exotic theories with the structure functions $U^\sigma_{\mu\nu}$ and V^σ_ν depending on momenta.
This extra term, of course, does not change the above proof of gauge invariance, because the source term is on-shell with $\delta S/\delta q = 0$.

12.2 t'Hooft trick and gauge breaking term

Delta function type gauge conditions in the Faddeev-Popov integral can be replaced by an additional term in the classical action, which is called gauge breaking term. This can be done according to the following t'Hooft trick. Using the gauge independence of the path integral we shift the gauge conditions functions χ^μ by a field independent quantity b^μ, $\chi^\mu \to \chi^\mu - b^\mu$, so that $Z_\chi = Z_{\chi-b}$ (in what follows we imply that we are on shell and omit the source argument of Z). Then, by noting that $Q_{\chi-b} = Q_\chi$, we insert in the integrand of the path integral for $Z_{\chi-b}$ the unity represented by the Gaussian integral over b^μ and take this integral by using

$\delta[\,\chi(g)-b\,]$. This leads to

$$1=\underbrace{\big(\,{\rm Det}\,c_{\mu\nu}[\,g\,]\,\big)^{1/2}\int Db\,e^{-\frac{i}{2}\,b^\mu\,c_{\mu\nu}[\,g\,]\,b^\nu}}$$

$$\Downarrow$$

$$\begin{aligned}
Z_{\chi-b}&=\int Dg\,\mu[\,g\,]\,\delta[\,\chi(g)-b\,]\,{\rm Det}\,Q_{\chi-b}[\,g\,]\,e^{\,i\,S[\,g\,]}\\
&=\int Dg\,\tilde\mu[\,g\,]\,{\rm Det}\,Q_\chi[\,g\,]\,e^{\,i\,\big(S[\,g\,]-\frac12\chi^\mu(g)c_{\mu\nu}[\,g\,]\chi^\nu(g)\big)}.
\end{aligned}\tag{12.14}$$

As a result we get instead of the delta function of gauge conditions extra term in the total *gauge-fixed* action $S_{\rm gf}$, which is quadratic in $\chi^\mu(g)$,

$$S_{\rm gf}=S-\frac12\,\chi^\mu c_{\mu\nu}\chi^\nu,\tag{12.15}$$

and additional modification of the integration measure $\mu[\,g\,]\to\tilde\mu[\,g\,]$,

$$\tilde\mu[\,g\,]=\big(\,{\rm Det}\,G_{ij}[\,g\,]\,{\rm Det}\,c_{\mu\nu}[\,g\,]\,\big)^{1/2}.\tag{12.16}$$

The invertible *gauge fixing matrix* $c_{\mu\nu}[\,g\,]$ is generally an arbitrary functional of the gauge field. When this matrix is ultralocal in time and space, $c_{\mu\nu}\mapsto c_{\mu\nu}(x)\delta(x_\mu-x_\nu)$, the measure $\tilde\mu[\,g\,]$ remains *local* and proportional to $\delta^{(4)}(0)$. However, in many cases, like higher-derivative gravity theory or Horava gravity, it is useful to take this functional matrix as a differential operator in spacetime.

12.3 Feynman-DeWitt-Faddeev-Popov ghosts

Fundamental step in the advancement of path integral method for gauge field models consists in the observation that the Faddeev-Popov functional determinant can be rewritten as a Gaussian integral over anti-commuting Grassmann fields C^μ and $\bar C_\nu$ with the algebra similar to the algebra of classical fermionic fields

$$C^\mu C^\nu=-C^\nu C^\mu,\quad \bar C_\mu\bar C_\nu=-\bar C_\nu\bar C_\mu,\quad \bar C_\mu C^\nu=-C^\nu\bar C_\mu.\tag{12.17}$$

By using the Gaussian integral (9.42)-(9.44) from the theory of bosonic-fermionic systems we see that the Faddeev-Popov determinant can be represented as

$${\rm Det}\,Q[\,g\,]=\int DC\,D\bar C\,e^{\,i\,\bar C_\mu Q^\mu_\nu[\,g\,]\,C^\nu}.\tag{12.18}$$

These Grassmann variables are called *Feynman-DeWitt-Faddeev-Popov ghost fields*. Therefore the path integral acquires the form of the integral over the full configuration space of gauge and ghost fields with the exponentiated total action,

$$Z=\int Dg\,DC\,D\bar C\,\tilde\mu[\,g\,]\,\exp i\,\Big(S-\frac12\,\chi^\mu c_{\mu\nu}\chi^\nu+\bar C_\mu Q^\mu_\nu\,C^\nu\Big).\tag{12.19}$$

The total action consists of the classical gauge invariant part, gauge breaking part and ghost action.

Why is this form useful? Point is that in certain class of gauge conditions ghost fields acquire dynamical properties similar to ordinary physical fields (except their statistics – despite their integer spin they are anti-commuting, and spin-statistics relation is violated, because the energy positivity requirements do not apply to ghost particles). This class of gauges is good because it can preserve Lorentz covariance and provides manifestly covariant perturbation and renormalization theory for gauge models. For this reason these gauges are called *relativistic*.

Let us consider several basic examples. We know from Lecture 5 examples of canonical gauge conditions on phase space variables, which allowed us to disentangle the physical sector. For QED it was the Coulomb gauge $\chi=\partial_iA^i$, for linearized gravity it was the set of gauges for spacetime diffeomorphisms $\chi^i=\partial_jh^{ij}$, $\chi_\perp=\delta_{ij}\pi^{ij}$. All these gauges generate ultralocal in time Faddeev-Popov operator $J^\mu_\nu\propto\delta(t_\mu-t_\nu)$. These gauges exclude all gauge and other unphysical modes of the theory except those of the physical sector. Only physical sector modes are propagating in these canonical gauges.

Relativistic gauges are different. In electrodynamics this is, for example, the Lorentz gauge $\chi = \partial_\mu A^\mu$. Under a special choice of gauge-fixing 1×1 "matrix" $c_{\mu\nu} = 1$, which is just a number, the gauge fixed action reads

$$S_{\rm gf}[\,A\,] = -\frac{1}{4}\int d^4x\, F_{\mu\nu}^2 - \frac{1}{2}\int d^4x\,(\partial_\mu A^\mu)^2 = -\frac{1}{2}\int d^4x\,(\partial_\mu A_\alpha)^2. \tag{12.20}$$

It describes the propagation of four components of the vector potential including the A_0-component and the unphysical spatially longitudinal component (A_0 is a ghost with a negative kinetic energy – here the term "ghost" is used not in the gauge theory sense, but rather indicating its instability due to negative kinetic energy).

Relativistic gauge in linearized Einstein theory is the harmonic (or linearized DeDonder or DeWitt) gauge $\chi^\mu = \partial_\nu h^{\mu\nu} - (1/2)\partial^\mu h$ which yields the gauge-fixed linearized gravitational action

$$S_{\rm gf}[\,h_{\mu\nu}\,] = S[\,h_{\mu\nu}\,] - \frac{1}{2}\int d^4x\,\chi^\mu\eta_{\mu\nu}\chi^\nu = -\frac{1}{4}\int d^4x\,(\eta^{\alpha\lambda}\eta^{\beta\sigma} + \eta^{\alpha\sigma}\eta^{\beta\lambda} - \eta^{\alpha\beta}\eta^{\lambda\sigma})\,\partial_\mu h_{\alpha\beta}\partial^\mu h_{\lambda\sigma}, \tag{12.21}$$

in which all ten metric fluctuations are dynamically propagating.

In both of the above cases relativistic gauge conditions contain Lagrange multipliers ($\lambda = A_0$ in electromagnetism and $\lambda^\mu \propto h^{0\mu}$ for linearized gravity) and their *time derivatives*. Generic structure of these gauge conditions $\chi^\mu = \chi^\mu(g,\dot g)$ is such that the matrix

$$a^\mu_\nu \equiv -\frac{\partial\chi^\mu}{\partial\dot\lambda^\nu}, \quad \det a^\mu_\nu \neq 0, \tag{12.22}$$

is non-degenerate. Their Faddeev-Popov operator is no longer ultralocal in time. Indeed, it is of the second order in time derivatives

$$\begin{aligned} Q^\mu_\nu &= \frac{\delta\chi^\mu}{\delta g^a}\,R^a_\nu = \frac{\delta\chi^\mu}{\delta\lambda^\alpha}\,R^\alpha_\nu + ... \\ &= \Big(\frac{\partial\chi^\mu}{\partial\dot\lambda^\alpha}\,\frac{d}{dt} + ...\Big)\Big(\delta^\alpha_\nu\,\frac{d}{dt} + ...\Big)\,\delta(t-t') + ... = \Big(-a^\mu_\nu\,\frac{d^2}{dt^2} + ...\Big)\,\delta(t-t'). \end{aligned} \tag{12.23}$$

With this operator the ghost action contains the kinetic term for ghost fields

$$\bar C_\mu Q^\mu_\nu\,C^\nu = \int dt\,\bar C_\mu\Big(-a^\mu_\nu\,\frac{d^2}{dt^2} + ...\Big)\,C^\nu = \int dt\,\Big(\dot{\bar C}_\mu\,a^\mu_\nu\,\dot C^\nu + ...\Big) \equiv \int dt\,L_{\rm ghost}\big(C,\bar C,\dot C,\dot{\bar C}\big). \tag{12.24}$$

Thus, all gauge and ghost fields $\varPhi = (\,g^a, C^\mu, \bar C_\mu\,)$ on equal footing enter the total action and its total Lagrangian

$$S_{\rm tot}[\,\varPhi\,] = S_{\rm gf}[\,g\,] + S_{\rm ghost}[\,g, C, \bar C\,] = \int dt\,L_{\rm tot}\big(\varPhi,\dot\varPhi\big), \tag{12.25}$$

and the Faddeev-Popov path integral (12.19) takes a very concise form

$$Z = \int D\varPhi\,\mu[\,\varPhi\,]\,e^{\,i\,S_{\rm tot}[\,\varPhi\,]}. \tag{12.26}$$

Interestingly, the local measure $\mu[\,\varPhi\,]$ also expresses via the (super)determinant of the Hessian matrix of the total Lagrangian with respect to velocities of boson-fermion variables $\dot\varPhi$

$$\mu[\,\varPhi\,] = \left[\,{\rm SDet}\left(\frac{\overrightarrow{\partial}}{\partial\dot\varPhi}\,L_{\rm tot}\,\frac{\overleftarrow{\partial}}{\partial\dot\varPhi}\right)\right]^{1/2} = \frac{\big({\rm Det}\,a_{ab}\big)^{1/2}}{{\rm Det}\,a^\mu_\nu}. \tag{12.27}$$

This follows from the block structure of the Hessian of the gauge-fixed Lagrangian

$$a_{ab} = \frac{\partial^2 L_{\rm gf}}{\partial\dot g^a\,\partial\dot g^b} = \begin{bmatrix} a_{ij} & a_{i\nu} \\ a_{\mu j} & a_{\mu\nu} \end{bmatrix}, \tag{12.28}$$

$$a_{ij} = G_{ij} - \frac{\partial\chi^\alpha}{\partial\dot q^i}\,c_{\alpha\beta}\,\frac{\partial\chi^\beta}{\partial\dot q^j}, \quad a_{i\mu} = \frac{\partial\chi^\alpha}{\partial\dot q^i}\,c_{\alpha\beta}\,a^\beta_\mu, \quad a_{\mu\nu} = -a^\alpha_\mu\,c_{\alpha\beta}\,a^\beta_\nu,. \tag{12.29}$$

(G_{ij} is defined by Eq.(11.31)) and its determinant

$$\det a_{ab} = \det G_{ij}\,\det c_{\alpha\beta}\,\big(\det a^\mu_\nu\big)^2, \tag{12.30}$$

which implies that the local measure (12.16) in the path integral of Eq.(12.19) equals the total measure (12.27) in the full gauge-ghost system of fields

$$\tilde\mu[\,g\,] = \big(\mathrm{Det}\,G_{ij}\,\mathrm{Det}\,c_{\alpha\beta}\big)^{1/2} = \frac{\big(\mathrm{Det}\,a_{ab}\big)^{1/2}}{\mathrm{Det}\,a^\mu_\nu} = \mu[\,\varPhi\,]. \tag{12.31}$$

Problem 12.3. Derive Eqs.(12.28)-(12.31)

Eqs.(12.26) and (12.27) show that in relativistic gauge all the fields become propagating and at the quantum level (that is in the path integral) are treated *nearly* on equal footing. "Nearly" means here that we for a while forget about the source term – for S-matrix no sources are turned on for ghost fields and for gauge fields g^a the source is weakly vanishing in the sense of Eq.(12.9). So, modulo what is at the external lines of Feynman diagrams (the lines that extend to $t=\pm\infty$) all the fields $\varPhi=(q^i,\lambda^\mu,C^\mu,\bar C_\mu)$ contribute to the path integral by the same rule – via the integral of exponentiated action and local measure.

The local measure for gauge fields and ghosts is also built by the same pattern – via the Hessian of the Lagrangian with respect to field velocities $\dot\varPhi$. The only difference is in statistics – the determinant of the matrix of the kinetic term for bosons (that is a_{ab} in Eq.(12.28)) and for Grassmann ghosts (that is $a^\mu_\nu = (\overrightarrow{\partial}/\partial\dot{\bar C}_\mu)\,L_{\rm ghost}\,(\overleftarrow{\partial}/\partial\dot C^\nu)$ in Eq(12.24)) go respectively into the numerator and denominator of the expression (12.31).

Note that in contrast to canonical gauges, in which all non-physical fields were excluded as a result of imposing the full set of constraints and auxiliary conditions (that is gauges), here all the fields are propagating. For ghost fields this was already shown above – the wave operator in their equations of motion (12.23) is of second order in time derivatives and demands imposing two boundary conditions per each ghost mode in order to specify its time evolution. The same holds for *all* gauge fields g^a – not only equations for q^i are dynamical, but the Lsagrange multipliers are dynamical as well – they acquire the kinetic term quadratic in $\dot\lambda^\mu$ from the gauge breaking term in the total action,

$$S_{\rm gf}[\,g\,] = \int dt\,\Big(\frac12\,\dot\lambda^\mu\,a_{\mu\nu}\,\dot\lambda^\nu + ...\Big), \tag{12.32}$$

where $a_{\mu\nu}$ is defined by Eq.(12.29). Natural question arises, how all these $n+m+2m$ dynamical variables, that is n variables q^i, m variables λ^μ, $2m$ ghosts C^μ and $\bar C_\mu$, $i=1,...n$, $\mu=1,...m$, can correspond to $n-m$ physical degrees of freedom within the canonical gauge fixing procedure.

$$\begin{array}{cccl} (q^i,\lambda^\mu)=g^a & \leftarrow & n+m & \text{variables}\\ C^\mu & \leftarrow & m & \text{variables}\\ \bar C_\mu & \leftarrow & m & \text{variables}\end{array}$$

Interpretation of this situation is that dynamical ghost fields of opposite statistics effectively subtract $2m$ degrees of freedom from the configuration space of original gauge fields g^a,

$$\#\,\mathrm{DoF} = n+m-2m = n-m. \tag{12.33}$$

This subtle mechanism of cancellation of quantum degrees of freedom is provided by the Feynman-DeWitt-Faddeev-Popov gauge fixing procedure of the above type. It takes place in the quantum interaction domain (or in the inner region of spacetime as opposed to the asymptotic future and past infinities corresponding to external lines of Feynman diagrams).

12.4 Boundary conditions for gauge and ghost fields

What has not yet been discussed yet is the imposition of boundary conditions on gauge and ghost fields in the Faddeev-Popov path integral. Implicitly, though, these boundary conditions have already been exploited in the proof of its gauge independence. Point is that, as we know now, the Faddeev-Popov operator Q^μ_ν in relativistic gauges is a differential operator, and its inversion requires the knowledge boundary conditions which would fix its Green's function uniquely. At the same time, when proving gauge independence, we used this Green's function and, moreover, varied it in Eq.(11.49) by using the variational equation from finite dimensional linear algebra,

$$\delta Q^{-1\mu}_{\;\;\;\nu} = -Q^{-1\mu}_{\;\;\;\alpha}\,\delta Q^\alpha_\beta\, Q^{-1\beta}_{\;\;\;\nu}.$$

However, not all possible boundary conditions for nonlocal Green's functions are consistent with this variational law.[12] Fortunately, the positive/negative frequency boundary conditions at $t\to\pm\infty$, which were derived for integration fields in the physical sector, are consistent with this variational law. Therefore, it is natural to assume that ghost fields and their Green's function $Q^{-1\mu}_{\;\;\;\nu}$ should satisfy the same Feynman boundary conditions as the physical modes.

We also have to specify boundary conditions for Lagrange multipliers and nonphysical modes among the full set of gauge fields $g^a = (q^i, \lambda^\mu)$. Consistency of the formalism requires that they all have to satisfy the same boundary conditions. Indeed, the proof of gauge independence uses the gauge transformation of g^a imitating the change of gauge conditions, $\Delta g^a = R^a_\mu\, Q^{-1\mu}_{\;\;\;\nu}\,\delta\chi^\nu$, and this transformation should not spoil the needed boundary conditions for g^a. Since R^a_μ is a quasi-local differential operator, Δg^a has the same boundary conditions at $t\to\pm\infty$ as the ghost Green's function $Q^{-1\mu}_{\;\;\;\nu}$, which confirms this property. Later we will derive tree-level Ward identities which establish a linear relation between the Green's function G^{ab} of g^a and the ghost Green's function $Q^{-1\mu}_{\;\;\;\nu}$, which will give additional justification of this statement.

12.5 Recovery of classical theory, initial conditions and counting physical degrees of freedom

Thus, at the quantum level the theory is described by the total action in which all gauge and ghost fields enter on equal footing,

$$S_{\rm tot}[\varPhi] = S[g] - \frac{1}{2}\,\chi^\mu[g]c_{\mu\nu}[g]\,\chi^\nu[g] + \bar C_\mu Q^\mu_\nu[g]\,C^\nu. \tag{12.34}$$

A natural question arises, how this action recovers the classical equations of motion which should hold in the tree-level approximation. This action generates on shell (that is without external sources) the variational equations which intertwine all the fields. In the simplified case when the gauge condition $\chi^\mu(g)$ is linear in the gauge field and $c_{\mu\nu}$ is a field-independent constant matrix these equations read

$$\frac{\delta S}{\delta g^a} - \frac{\delta\chi^\mu}{\delta g^a}\,c_{\mu\nu}\chi^\nu + \bar C_\mu\,\frac{\delta Q^\mu_\nu}{\delta g^a}\,C^\nu = 0, \tag{12.35}$$

$$Q^\mu_\nu\,C^\nu = 0, \tag{12.36}$$

$$Q^\mu_\nu\,\bar C_\mu = 0. \tag{12.37}$$

Homogeneity of linear equations for ghost fields should not create an illusion that they should automatically vanish, because the matrix $Q^\mu_\nu = Q^\mu_\nu(d/dt)$ and its functionally transposed one in (12.37) are second order differential operators which have zero modes – propagating modes of ghost fields. To rule them out we impose, as was discussed above, positive/negative frequency boundary conditions at infinity, so that classically they indeed vanish. Particles of ghost fields are not physical, and we do not include them into the initial and final states of scattering amplitudes. With $C^\mu = \bar C_\mu = 0$ we get the equation for gauge fields

$$\frac{\delta S}{\delta g^a} - \frac{\delta\chi^\mu}{\delta g^a}\,c_{\mu\nu}\chi^\nu = 0, \tag{12.38}$$

which is "spoiled" by the gauge breaking term – differs from the classical equation $\delta S/\delta g^a = 0$. We know that the classical action is gauge invariant, that is it satisfies the Noether identity which is valid for *all* values of

[12] For example, retarded and advanced Green's functions $G^\pm$ of the second order operator F, $FG^\pm = -1$, satisfy this law $\delta G^\pm = G^\pm\,\delta F\,G^\pm$, as can be easily checked by varying this equation and comparing boundary conditions on both sides of equations. But the Green's function $G = (G^+ + G^-)/2$ fails to satisfy it.

gauge fields

$$\frac{\delta S}{\delta g^a}\, R^a_\mu = 0, \tag{12.39}$$

so that any solution of $\delta S/\delta g^a = 0$ is unique only up to a gauge transformation – if g_0^a is some solution, then $g_0^a + R^a_\mu(g_0) f^\mu$ is also a solution with a generic function of time $f^\mu = f^\mu(t)$. Educated guess hints that Eq.(12.38) would correspond to the classical equation of motion in the gauge $\chi^\mu(g) = 0$ which is supposed to rule out this ambiguity. Question is, how to enforce the second gauge breaking term to vanish and thus pick up the solution in this concrete gauge. This is done as follows. Contract Eq.(12.38) with the the gauge generator R^a_α and use the Noether identity (12.39). This immediately leads to

$$Q^\mu_\alpha\, c_{\mu\nu}\chi^\nu = 0. \tag{12.40}$$

which in view of (12.23) is a homogeneous second order equation on the gauge conditions functions

$$Q^\mu_\alpha(d/dt)\, c_{\mu\nu}\chi^\nu = \Big(-\frac{d^2}{dt^2}\, a^\mu_\alpha + ...\Big)\, c_{\mu\nu}\chi^\nu = 0. \tag{12.41}$$

For its solution to be zero *at all moments of time* this equation should have zero initial data at some initial moment of time t_- for both χ^μ and $\dot\chi^\mu$. Let us see what does this mean from the viewpoint of counting the number of physical degrees of freedom.

Equations (12.38) in the relativistic gauge form a set of second order equations for all g^a

$$\frac{\delta S}{\delta g^a} - \frac{\delta \chi^\mu}{\delta g^a}\, c_{\mu\nu}\chi^\nu = -a_{ab}\,\ddot g^b + ... = 0 \tag{12.42}$$

with a non-degenerate matrix a_{ab}, (12.28)-(12.29), where the second order derivatives of Lagrange multipliers $g^\mu = \lambda^\mu$ are contributed from the gauge breaking term. Therefore initial conditions for this equation consist of $2(n+m)$ initial values of g^a and $\dot g^a$ at t_-. Subtraction of $2m$ – the number of conditions $\chi^\mu\,|_{t_-} = 0$ and $\dot\chi^\mu\,|_{t_-} = 0$ – brings us to $2n$ independent initial data[13], which is still too much compared to the number $2(n-m)$ anticipated from the canonical formalism with a unitary gauge. However, there are also *residual gauge transformations*. Remember that relativistic gauge conditions *do not* fully fix the freedom in gauge transformations. These are those $\Delta^v g^a = R^a_\mu v^\mu$ with special gauge parameters v^μ which preserve the gauge conditions and solve the equations

$$\Delta^v \chi^\mu = Q^\mu_\nu(d/dt) v^\nu(t) = 0. \tag{12.43}$$

Since these are second order differential equations in time, their solutions are parameterized by $2m$ initial data, v and $\dot v$ at t_-. This allows one to freely change more $2m$ initial conditions or reduce the number of physically different initial conditions by extra $2m$. Therefore, equation (12.42) plus initial conditions for gauge functions $\chi^\mu\,|_{t_-} = 0$ and $\dot\chi^\mu\,|_{t_-} = 0$ imply the overall counting of physical degrees of freedom which coincides with the canonical one

$$2(n+m) - 2m - 2m = 2(n-m)\,. \tag{12.44}$$

Problem 12.4. Why in this counting we do not count nondynamical equations – first class constraints, $\delta S/\delta\lambda^\mu = -T_\mu(g,\dot g) = 0$, as an additional set of m restrictions on the initial data?
Hint. Note that these constraints are no longer independent and follow from the already counted restrictions $\chi, \dot\chi\,|_{t_-} = 0$ in view of equations of motion (12.42)

$$\frac{\delta S}{\delta\lambda^\mu} = -\frac{\delta\chi^\alpha}{\delta\lambda^\mu}\, c_{\alpha\beta}\chi^\beta \propto \chi, \dot\chi.$$

[13] The time derivative $\dot\chi^\mu$ contains $\ddot\lambda^\mu$, but it can be expressed via equations of motion in terms of g^a and $\dot g^a$, so that $\dot\chi^\mu = 0$ forms m restrictions on the inital g and $\dot g$.

Problem 12.5. Show that the result of the degrees of freedom counting is the same if we just consider the system of original (non gauge-fixed) equations of motion plus relativistic gauge conditions

$$\left\{ \begin{array}{l} \delta S/\delta g^a = 0, \\ \chi^\mu = -a^\mu_\nu \dot\lambda^\nu + ... = 0 \end{array} \right. .$$

Note that in this case the number of first class constraints actually participates in the counting.

Lecture 13. Tree-level and one-loop approximations, Ward identities, BRST symmetry

- Tree-level and one-loop approximations
- Ward identities for bare vertices
- BRST symmetry of the Faddeev-Popov path integral
- Ward-Slavnov identities
- Zinn-Justin equation for the effective action

13.1 Tree-level and one-loop approximations

Expand the action in the off-shell generating functional (12.26)

$$Z[\,J\,] = \int D\Phi\,\mu[\,\Phi\,]\,e^{\,i\,S_{\rm tot}[\,\Phi\,]+iJ_a g^a} \tag{13.1}$$

up to quadratic order in quantum perturbations around the stationary point $\Phi_0 = (g_0, C_0, \bar C_0)$ of $S_{\rm tot}[\,\Phi\,] + J_a g^a$ and calculate it in the the-level and one-loop approximation. Here the sources to ghost fields are not turned on for reasons discussed above – absence of ghost particles in the external lines of scattering amplitudes. Therefore, ghosts at the stationary point vanish, $C_0 = 0, \bar C_0 = 0$, whereas g_0 is an off-shell solution of the equation of motion with the gauge-fixed action (12.15),

$$\frac{\delta S_{\rm gf}}{\delta g^a}\Big|_{\,g=g_0} = \left[\,\frac{\delta S}{\delta g^a} - \frac{\delta\chi^\mu}{\delta g^a}\,c_{\mu\nu}\chi^\nu\,\right]_{\,g=g_0} = -J_a \tag{13.2}$$

Then the answer for the background field functional $\bar W[\,g_0\,]$ or the effective action $\varGamma[\,\langle g\rangle\,]$, where $\langle g\rangle$ is a notation for the mean gauge field defined in analogy with (10.11)

$$\langle\, g^a\rangle = \frac{1}{Z[\,J\,]}\,\frac{\delta Z[\,J\,]}{i\delta J_a}\Big|_{\,J\neq 0} = \frac{\delta W[\,J\,]}{\delta J_a}, \tag{13.3}$$

is given by the set of diagrams of Lecture 9. In these diagrams the loops are formed by the propagators of both gauge and ghost fields. The gauge field propagator is the Green's function of the Hessian of the gauge-fixed action

$$F_{ab} = \frac{\delta^2 S_{\rm gf}}{\delta g^a \delta g^b} = \frac{\delta^2 S}{\delta g^a \delta g^b} - \frac{\delta\chi^\mu}{\delta g^a} c_{\mu\nu} \frac{\delta\chi^\nu}{\delta g^b} \tag{13.4}$$

(we again consider the most convenient case of gauge functions χ^μ linear in the field and the constant gauge-fixing matrix $c_{\mu\nu}$) and the ghost propagator is the Green's function $Q^{-1\mu}_{\ \ \nu}$. Both gauge and ghost operators are nontrivial functionals of the gauge field, $F_{ab}[\,g\,]$ and $Q^\mu_\nu[\,g\,]$, where g is $g_0[\,J\,]$ and $\langle\, g^a\rangle$ respectively for $\bar W[\,g_0\,]$ and $\varGamma[\,\langle g\rangle\,]$.

Note that the operator (13.4), is nondegenerate including the on-shell configurations where the Hessian of the classical action $S_{ab} = \delta^2 S/\delta g^a \delta g^b$ has zero modes and not invertible. The latter property follows from functional differentiation of the Noether identity (12.39)

$$R^a_\mu \frac{\delta^2 S}{\delta g^a \delta g^b} + \frac{\delta S}{\delta g^a}\frac{\delta R^a_\mu}{\delta g^b} = 0, \tag{13.5}$$

which means that on shell the gauge generators are zero eigenvalue eigenfunctions of S_{ab},

$$R^a_\mu \frac{\delta^2 S}{\delta g^a \delta g^b}\bigg|_{\delta S/\delta g=0} = 0. \tag{13.6}$$

The gauge breaking term of (13.4) makes this operator invertible and having a well-defined Green's function G^{ab}

$$F_{ab} G^{bc} = -\delta^c_a. \tag{13.7}$$

With all this at hand we can write down the answer for the effective action in the one-loop approximation

$$\varGamma[\,g\,] = S[\,g\,] - \frac{1}{2}\,\chi^\mu[\,g\,]\,c_{\mu\nu}\chi^\mu[\,g\,] + \hbar\varGamma_{\rm one-loop}[\,g\,] + O(\hbar^2), \tag{13.8}$$

$$\varGamma_{\rm one-loop}[\,g\,] = -\frac{1}{2i}\,{\rm Tr}\,\ln F_{ab}[\,g\,] + \frac{1}{i}\,{\rm Tr}\,\ln Q^\mu_\nu[\,g\,]. \tag{13.9}$$

Problem 13.1. Reproduce Eqs.(13.8)-(13.9)

Here for brevity we omit brackets in the notation for mean field. Also we disregard the contribution of local measure $-(i/2){\rm Tr}\,\ln a_{ab} + i\,{\rm Tr}\,\ln a^\mu_\nu \propto \delta(0)$ responsible for cancellation of strongest volume divergences.[14]

13.2 Ward identities for bare vertices

Let us explicitly demonstrate on-shell gauge independence of effective action in the one-loop approximation – the basic property of the Faddeev-Popov path integral. In the tree-level approximation this property is trivial. Indeed, the on-shell condition

$$\frac{\delta\varGamma[\langle g\rangle]}{\delta\langle\, g^a\rangle} = 0, \tag{13.10}$$

for the tree-level effective action $\varGamma_{\rm tree} = S_{\rm gf} + O(\hbar)$ reads as Eq.(12.38) which after contraction with the gauge generator results in (12.39). In view of boundary conditions at asymptotic infinity gauge conditions are enforced throughout the whole spacetime bulk, so that gauge breaking term in $S_{\rm gf}$ vanishes and $\varGamma_{\rm tree} = S$, which is obviously gauge independent.

In the one-loop approximation gauge independence of $\varGamma$ follows from Ward identities for bare propagators. To derive them contract the equation for gauge Green's function (13.7) with the gauge generator,

$$R^a_\alpha\left(S_{ab} - \chi^\mu_a c_{\mu\nu}\chi^\nu_b\right) G^{bc} = -R^c_\alpha, \tag{13.11}$$

and use the identity (13.6) to show that

$$Q^\mu_\alpha\, c_{\mu\nu}\chi^\nu_b\, G^{bc}\bigg|_{\delta S/\delta g=0} = R^c_\alpha, \tag{13.12}$$

where we introduced the abbreviation

$$\chi^\mu_a \equiv \frac{\delta\chi^\mu}{\delta g^a}. \tag{13.13}$$

Then, since Q^μ_α is invertible we have on shell the following relation between the gauge and ghost propagators

$$c_{\mu\nu}\chi^\nu_b\, G^{bc}\bigg|_{\delta S/\delta g=0} = Q^{-1}{}^\alpha_\mu\, R^c_\alpha\bigg|_{\delta S/\delta g=0}. \tag{13.14}$$

[14] With the inclusion of local measure the one-loop answer retains the same structure, but with the *forms* F_{ab} and Q^μ_ν replaced respectively by the *operators* $F^a_b = (a^{-1})^{ac}F_{cb} = -\delta^a_b d^2/dt^2 + \ldots$ and $(a^{-1})^\mu_\alpha Q^\alpha_\nu = -\delta^\mu_\nu d^2/dt^2 + \ldots$ with unit coefficients in kinetic terms.

This identity is usually known as a statement that the longitudinal part of the gauge propagator is given by the ghost propagator. Consider the example from the theory of electromagnetic field (12.20), but with another choice of the gauge fixing parameter – 1×1 "matrix" $c_{\mu\nu}\mapsto 1/2\alpha$,

$$S_{\rm gf}[\,A\,]=-\frac{1}{4}\int d^4x\,F_{\mu\nu}^2-\frac{1}{2\alpha}\int d^4x\,(\partial_\mu A^\mu)^2. \tag{13.15}$$

We have the following set of corresponding relations

$$g^a\mapsto A_\mu,\ R^a_\alpha f^\alpha\mapsto\partial_\mu f,\ \chi^\mu\mapsto\chi=\partial_\alpha A^\alpha,\ S_{ab}\mapsto F^{\mu\nu}=\Box\eta^{\mu\nu}-\partial^\mu\partial^\nu, \tag{13.16}$$

$$G^{ab}\mapsto G_{\mu\nu}=-\frac{1}{\Box}\Big[\eta_{\mu\nu}-(1-\alpha)\frac{\partial_\mu\partial_\nu}{\Box}\Big],\ Q^{-1}{}^{\alpha}_{\mu}\mapsto Q^{-1}=\frac{1}{\Box}, \tag{13.17}$$

$$\chi^\mu_a\mapsto\frac{\delta\chi(x)}{\delta A_\mu(y)}=\partial^\mu\delta(x,y), \tag{13.18}$$

$$c_{\mu\nu}\chi^\nu_b\,G^{bc}\mapsto\frac{1}{\alpha}\,\partial^\mu G_{\mu\nu}=\frac{1}{\Box}\,\partial_\nu\mapsto Q^{-1}{}^{\alpha}_{\mu}\,R^c_\alpha \tag{13.19}$$

Problem 13.2. Check these relations and interpret the meaning of the limit $\alpha=0$ in the expression for the propagator.

Now we are ready to prove on-shell gauge independence of the one-loop effective action. Perform in Eq.(13.9) infinitesimal variation of the gauge matrix χ^μ_a and use (13.14)

$$\begin{aligned}i\delta_\chi\varGamma_{\rm one-loop}[\,g\,]&=-\frac{1}{2}\,\delta_\chi{\rm Tr}\,\ln F_{ab}[\,g\,]+\delta_\chi{\rm Tr}\,\ln Q^\mu_\nu[\,g\,]\\&=\Big[-G^{ab}\chi^\mu_b\,c_{\mu\nu}+Q^{-1}{}^{\mu}_{\nu}\,R^a_\mu\Big]\,\delta\chi^\nu_a\propto\frac{\delta S}{\delta g}.\end{aligned} \tag{13.20}$$

On-shell condition (13.10) in the one-loop order should be retained up to terms $O(\hbar^2)$, $\delta_\chi\varGamma_{\rm one-loop}\propto\hbar\delta S/\delta g=\hbar\delta\varGamma/\delta g+O(\hbar^2)$, so that it reduces to $\delta S/\delta g=0$. Therefore the above result indeed implies gauge independence of the effective action.

Problem 13.3. Generalize Eq.(13.14) to off-shell configurations with $\delta S/\delta g\neq 0$ and similarly to (13.20) prove that one-loop effective action is independent of the choice of the gauge-fixing matrix $c_{\mu\nu}$.

13.3 BRST symmetry of the Faddeev-Popov path integral

Great advantage of the Feynman-DeWitt-Faddeev-Popov path integral is that its quantum integration measure, which follows from gauge fixing procedure, has a representation in terms of additional *ghost* integration variables whose Lagrangian is local and manifestly covariant in spacetime. Renormalization of ultraviolet divergences by local counterterms critically depends on local nature of all fields – both physical and unphysical. This statement is a part of Bogolyubov-Parasyuk-Hepp-Zimmerman (BPHZ) theory, which goes beyond this course and will be considered later. Without the ghost fields representation of the quantum measure application of BPHZ theory in gauge-invariant models would be impossible. Therefore, even though particles of ghost fields do not appear in physical states of scattering amplitudes, which was the reason why we included the sources in (13.1) only for gauge fields g^a, the studies of the ultraviolet renormalization of gauge models requires to treat gauge and ghost fields on equal footing. It turns out that this treatment reveals another symmetry which plays a cruical role for renormalization of gauge theories. This is the Becchi-Rouet-Stora-Tyutin symmetry which we will study now and derive the Ward-Slavnov-Taylor identities for the generating functional and Zinn-Justin equation for effective action underlying this renormalization. We restrict ourselves in this course with the class of theories with closed algebra of gauge transformations. Extension beyond this class to open algebras also exists in the canonical framework, known as Batalin-Fradkin-Vilkovisky (BFV) formalism, and in the Lagrangian framework known as Batalin-Vilkovisky (BV) theory.

We begin by repeating that the action of the theory $S[g]$ is invariant under local gauge transformations with the generators R^a_α forming the linear infinite-dimensional representation of the group and satisfying the Lie algebra (11.46) with *structure constants* $C^\gamma_{\alpha\beta}$

$$R^a_\alpha \frac{\delta S}{\delta g^a} = 0, \tag{13.21}$$

$$R^b_\alpha \frac{\delta R^a_\beta}{\delta g^b} - R^b_\beta \frac{\delta R^a_\alpha}{\delta g^b} = C^\gamma_{\alpha\beta} R^a_\gamma. \tag{13.22}$$

The generators R^a_α are *linear* local functions of the gauge fields as in all the examples considered above, and the structure constants of their gauge algebra satisfy a standard Jacobi identity.

$$C^\alpha{}_{\beta[\gamma} C^\beta{}_{\lambda\mu]} = 0\,, \tag{13.23}$$

where square brackets mean total anisymmetrization over three indices inside them. As everywhere above we assume that gauge fields g^a are bosonic, the parameter f^α of the gauge transformation $\Delta^f g$ is also bosonic, so that Faddeev-Popov ghosts are Grassmann variables

$$\epsilon(g^a) \equiv \epsilon_a = 0, \quad \epsilon(f^\alpha) \equiv \epsilon_\alpha = 0, \quad \epsilon(C^\alpha) = \epsilon(\bar{C}_\alpha) = \epsilon_\alpha + 1 = 1. \tag{13.24}$$

Let us now make in the generating functional (12.19) the transformation opposite to the t'Hooft trick – represent the gauge breaking term in the form of the Gaussian path integral over the auxiliary field b_α which we will call the Lagrange multiplier for gauge conditions functions

$$e^{\frac{i}{2}\chi^\alpha c_{\alpha\beta}\chi^\beta} \big(\mathrm{Det} c_{\alpha\beta}\big)^{1/2} = \int Db\, e^{i\left(b_\alpha\chi^\alpha - \frac{1}{2}c^{\alpha\beta}b_\alpha b_\beta\right)}, \quad c^{\alpha\beta} = \big(c_{\alpha\beta}\big)^{-1}. \tag{13.25}$$

Then, if we extend the full set of quantum fields by this extra boson variable,

$$\Phi^I = g^a, C^\alpha, \bar{C}_\alpha \to \Phi^I = g^a, C^\alpha, \bar{C}_\alpha, b_\alpha, \quad \epsilon_b = 0, \tag{13.26}$$

then the generating functional takes the form of the path integral

$$e^{iW[J]} = \int D\Phi\, e^{i\Sigma[\Phi] + iJ_a g^a}, \tag{13.27}$$

where the new action of boson-fermion set of fields has the form

$$\Sigma[\Phi] = S[g] + b_\alpha\Big(\chi^\alpha(g) - \frac{1}{2}c^{\alpha\beta}b_\beta\Big) - \bar{C}_\alpha\left(\chi^\alpha_a R^a_\beta\right)C^\beta \tag{13.28}$$

Consider now the following *Grassmann* parity BRST transformation on the space of fields (13.26) – here Grassmann means that this transformation by the fermionic *BRS operator* $\boldsymbol{s}$,

$$\boldsymbol{s} = \big(\boldsymbol{s}\Phi^I\big)\frac{\delta}{\delta\Phi^I}, \tag{13.29}$$

converts the bosonic variable into some other Grassmann parity variable and vice versa, $\epsilon(\boldsymbol{s}\Phi) = \epsilon(\Phi) + 1$.[15] The BRST transformation of the original gauge field coincides with its gauge transformation in which the gauge parameter is identified with the ghost field, $f^\alpha \mapsto C^\alpha$, and the full list of transformations $\boldsymbol{s}\Phi^I$ reads

$$\boldsymbol{s}g^a = \Delta^C g^a = R^a_\alpha C^\alpha, \tag{13.30}$$

$$\boldsymbol{s}C^\alpha = \frac{1}{2}C^\alpha_{\gamma\beta}C^\beta C^\gamma, \quad \boldsymbol{s}\bar{C}_\alpha = b_\alpha, \quad \boldsymbol{s}b_\alpha = 0\,. \tag{13.31}$$

This transformation is nilpotent,

$$\boldsymbol{s}^2 = 0, \tag{13.32}$$

[15] Otherwise, it maybe regarded as the transformation of the field $\delta^\varepsilon\Phi = \varepsilon\boldsymbol{s}\Phi$ with an infinitesimal *global* anti-commuting parameter ε. Therefore, the BRST transformation is not local, but rather global transformation with a Grassmann parameter constant in spacetime.

as can be seen by applying again the action of $\boldsymbol{s}$ and using the Leibnitz rule. For example, acting by $\boldsymbol{s}$ on (13.29) one has

$$\boldsymbol{s}^2 g^a = \big(\boldsymbol{s} g^b\big)\, \frac{\delta R^a_\alpha}{\delta g^b}\, C^\alpha + R^a_\alpha \big(\boldsymbol{s} C^\alpha\big), \tag{13.33}$$

which vanishes in virtue of (13.22). Similarly, $\boldsymbol{s}^2 C^\alpha = 0$ in virtue of the Jacobi identity (13.23) and so on.

In terms of this BRST operator the full action (13.28) can be rewritten as

$$\varSigma[\varPhi] = S[g] + \boldsymbol{s}\,\varPsi[\varPhi], \tag{13.34}$$

where $\varPsi[\varPhi]$ is the so-called *gauge fermion* – Grassmann parity object, $\epsilon(\varPsi) = 1$, containing all gauge-fixing information, that is the gauge conditions $\chi^\alpha(g) = \chi^\alpha_a g^a$ and gauge-fixing matrix $c^{\alpha\beta}$,

$$\varPsi[\varPhi] = \bar{C}_\alpha \Big(\chi^\alpha(g) - \frac{1}{2}\, c^{\alpha\beta}\, b_\beta\Big). \tag{13.35}$$

In this form the full action $\varSigma[\varPhi]$ is explicitly BRST-invariant under the transformations (13.30)-(13.31)

$$\boldsymbol{s}\varSigma[\varPhi] = 0, \tag{13.36}$$

because the first classical term in (13.34) is gauge invariant and the second *BRS-exact* term is identically annihilated by the action of the nilpotent operator $\boldsymbol{s}$ for any choice of $\varPsi[\varPhi]$.

13.4 Ward-Slavnov identities

Our next task is to derive the consequences of this BRST-invariance for the generating functional $Z = \exp(iW)$ and for the effective action $\varGamma$ of the theory. These consequences are Ward-Slavnov-Taylor identities which are special functional differential equations in variational derivatives with respect to the sources and mean field – the arguments of W and $\varGamma$. These equations underlie the renormalization properties of gauge theories, their counterterms and relations between their various Green's functions. To make their analysis manageable these equations should be simple enough, that is not higher than of the first order in functional derivatives. An obvious difficulty with this requirement is that the BRST transformations (13.30)-(13.31) are nonlinear in fields. For example, the transformation (13.30) of g^a is bilinear in g and C, because the generator R^a_α is linear in the field, and so is the ghost field transformation. This means that relevant terms in these identities canot be generated by first-order functional derivatives in sources dual to the fields $\varPhi$.

To circumvent this difficulty, let us introduce sources not only for the basic set of quantum fields $\varPhi^I$, but also to their BRST-transformations $\boldsymbol{s}\varPhi^I$. To begin with, it suffices to introduce the sources J_a, $\bar{\xi}_\alpha$, ξ^α, y^α dual respectively to g^a, C^α, $\bar{C}_\alpha$, b_α and the sources γ_a, ζ_α for the *BRST-transformations* of the fields g^a and C^α belonging to the so-called *minimal* sector. These sources, of course, have the same statistics as their duals.

Thus we have the full set of the sources,

$$\mathcal{J} = J_a,\, \bar{\xi}_\alpha,\, \xi^\alpha,\, y^\alpha,\, \gamma_a,\, \zeta_\alpha, \quad \epsilon(J_a) = \epsilon(y^\alpha) = \epsilon(\zeta_\alpha) = 0, \quad \epsilon(\xi^\alpha) = \epsilon(\bar{\xi}_\alpha) = \epsilon(\gamma_a) = 1, \tag{13.37}$$

in the generating functional

$$e^{\,i\,W[\mathcal{J}\,]} = \int D\varPhi\, \exp\, i\,\Big\{\varSigma[\varPhi] + J_a g^a + \bar{\xi}_\alpha C^\alpha + \xi^\alpha \bar{C}_\alpha + y^\alpha b_\alpha + \gamma_a\,(\boldsymbol{s} g^a) + \zeta_\alpha(\boldsymbol{s} C^\alpha)\Big\}. \tag{13.38}$$

Functional differentiations with respect to these sources will generate the correlators of their duals.

Now, it is easy to see that a long tail of source terms here can be absorbed into the modified *BRS-exact* term of the action by a simple generalization of the BRS-operator and gauge fermion. For this purpose let us include the sources J_a, $\bar{\xi}_\alpha$, ξ^α into the following *extended* BRS-operator,

$$\boldsymbol{s} \to \boldsymbol{Q} = \boldsymbol{s} - J_a \frac{\delta}{\delta\gamma_a} + \bar{\xi}_\alpha \frac{\delta}{\delta\zeta_\alpha} + \xi^\alpha \frac{\delta}{\delta y^\alpha}, \qquad \boldsymbol{Q}^2 = 0, \tag{13.39}$$

and the sources γ_a and ζ_α and y^α – into the *extended* gauge fermion

$$\varPsi[\varPhi] \to \boldsymbol{\varPsi}[\varPhi, \gamma, \zeta, y\,] \equiv \varPsi[\varPhi] - \gamma_a g^a + \zeta_\alpha C^\alpha + y^\alpha \bar{C}_\alpha. \tag{13.40}$$

Note that this new BRS-operator is also nilpotent as it follows from the Grassmann parity of ghost fields and anti-commutativety of their derivatives.

Problem 13.4. Prove that $\boldsymbol{Q}^2 = 0$

Then, non-classical part of the exponential in the integrand of (13.38) again takes the BRS-exact form with this new BRS-operator. Indeed, the action of $\boldsymbol{Q}$ on $\boldsymbol{\Psi}$ easily yields

$$\boldsymbol{Q}\,\boldsymbol{\Psi} = \boldsymbol{s}\,\Psi[\,\Phi\,] + J_a g^a + \bar{\xi}_\alpha C^\alpha + \xi^\alpha \bar{C}_\alpha + y^\alpha b_\alpha + \gamma_a\,(\boldsymbol{s}g^a) + \zeta_\alpha(\boldsymbol{s}C^\alpha)\,. \tag{13.41}$$

Thus, finally

$$e^{\,i\,W[\,\mathcal{J}\,]} = \int D\Phi\, e^{\,i\boldsymbol{\Sigma}[\,\Phi,\mathcal{J}\,]}, \tag{13.42}$$

$$\boldsymbol{\Sigma}[\,\Phi,\mathcal{J}\,] = S[\,g\,] + \boldsymbol{Q}\,\boldsymbol{\Psi}[\,\Phi,\mathcal{J}\,], \tag{13.43}$$

where the new BRST action is obviously BRS-invariant with respect to the extended BRS-operator *depending on sources*

$$\boldsymbol{Q}\boldsymbol{\Sigma}[\,\Phi,\mathcal{J}\,] = 0. \tag{13.44}$$

Then the derivation of Ward identities is straightforward. We have

$$\begin{aligned} 0 = \int D\Phi\,\boldsymbol{Q}\,e^{\,i\boldsymbol{\Sigma}} &= \int D\Phi\,\Big(\boldsymbol{s} - J\frac{\delta}{\delta\gamma} + \bar{\xi}\frac{\delta}{\delta\zeta} + \xi\frac{\delta}{\delta y}\Big)\,e^{\,i\boldsymbol{\Sigma}} \\ &= \Big(-J\frac{\delta}{\delta\gamma} + \bar{\xi}\frac{\delta}{\delta\zeta} + \xi\frac{\delta}{\delta y}\Big)\,e^{\,iW} + \int D\Phi\,\boldsymbol{s}\,e^{\,i\boldsymbol{\Sigma}}, \end{aligned} \tag{13.45}$$

where the variables inside the parenthesis do not participate in integration (and therefore can be extracted from under the sign of integration) and the last term can be transformed via functional integration by parts as follows,

$$\int D\Phi\,\boldsymbol{s}\,e^{\,i\boldsymbol{\Sigma}} \equiv \int D\Phi\,(\boldsymbol{s}\Phi^I)\frac{\delta}{\delta\Phi^I}e^{\,i\boldsymbol{\Sigma}} = -\int D\Phi\,\Big(\frac{\delta}{\delta\Phi^I}\,\boldsymbol{s}\Phi^I(\Phi)\Big)\,e^{\,i\boldsymbol{\Sigma}}, \tag{13.46}$$

and either disregarded since it is proportional to $\delta(0)$ or cancelled by the local measure $\mu[\,g\,]$, whose contribution we were omitting for brevity throughout the above derivation (if included back, $D\Phi \to D\Phi\mu[\,g\,]$). Cancellation takes place due to its transformation (11.53),

$$\boldsymbol{s}\mu[\,g\,] = -\mu[\,g\,]\left[\frac{\delta R^a_\alpha}{\delta g^a} + C^\beta_{\alpha\beta}\right]C^\alpha. \tag{13.47}$$

Therefore, equation (13.45) gives a final rather simple form of Ward identities for the generating functional of connected diagrams

$$\left[-\,J_a\frac{\delta}{\delta\gamma_a} + \bar{\xi}_\alpha\frac{\delta}{\delta\zeta_\alpha} + \xi^\alpha\frac{\delta}{\delta y^\alpha}\right]W[\,\mathcal{J}\,] = 0. \tag{13.48}$$

To remember the structure of the differential operator here note that it is built by the following rule. The rule is the contraction of the sources dual to quantum fields $(g^a, C^\alpha, \bar{C}_\alpha)$ with the functional derivatives with respect to the sources *dual to the BRST transformations of these fields* $(\boldsymbol{s}g^a, \boldsymbol{s}C^\alpha, \boldsymbol{s}\bar{C}_\alpha)$ – the last one $\boldsymbol{s}\bar{C}_\alpha = b_\alpha$ and its source is y^α.

There is another important equation for $W[\,\mathcal{J}\,]$ which follows from the equation of motion for the Lagrange multiplier b_α. It reads

$$\int D\Phi\,\frac{\delta}{i\,\delta b_\alpha}\,e^{\,i\boldsymbol{\Sigma}} = \left[\chi^\alpha_a\frac{\delta}{\delta J_a} - c^{\alpha\beta}\frac{\delta}{\delta y^\beta} + y^\alpha\right]e^{\,iW[\,\mathcal{J}\,]} = 0, \tag{13.49}$$

or

$$\left[\chi^\alpha_a \frac{\delta}{\delta J_a} - c^{\alpha\beta}\frac{\delta}{\delta y^\beta}\right] W[\mathcal{J}\,] + y^\alpha \; = 0, \tag{13.50}$$

where we took into account Eqs.(13.28) and (13.38) and generated the linear field dependence of the pre-exponential factor in the integrand by the first order functional derivatives $\delta/\delta J_a$ and $\delta/\delta y^\beta$. Note that the last derivation essentially relies on the property that the gauge-fixing conditions are linear in the quantum field and the gauge-fixing matrix $c^{\alpha\beta}$ is field independent.

13.5 Zinn-Justin equation for the effective action

Ward identities derived above can be reformulated in terms of the effective action $\varGamma$ – the generating functional of one-particle irreducible diagrams or full quantum vertices of the theory. This functional is important because it actually serves as a generalization of the classical action to the quantum level, it generates the effective equations of motion for the mean field $\langle\, g^a\,\rangle$ – the argument of $\varGamma[\,\langle\, g^a\,\rangle\,]$ – and its UV divergent part gives the counterterms which renormalize divergent scattering amplitudes and quantum expectation values.

As was discussed in Lecture 9, effective action follows from the generating functional of connected graphs by the Legendre transform (10.12) with respect to the sources. In the case of gauge field theory we will assume that "extra" sources γ_a and ζ_α do not participate in the Legendre transform $W[\,\mathcal{J},\gamma,\zeta\,] \to \varGamma[\,\langle\,\varPhi\,\rangle,\gamma,\zeta\,]$ which only parametrically depends on them,

$$\varGamma[\,\langle\varPhi\rangle,\gamma,\zeta\,] = W[\,\mathcal{J},\gamma,\zeta\,] - \mathcal{J}_I\langle\,\varPhi^I\rangle, \quad \mathcal{J}_I = (J_a,\bar\xi_\alpha,\xi^\alpha,y^\alpha). \tag{13.51}$$

Here the sources $\mathcal{J} = \mathcal{J}[\,\langle\varPhi\rangle,\gamma,\zeta\,]$ are expressed as functionals of the mean fields and extra sources from the equations

$$\langle\,\varPhi^I\rangle = \frac{\delta}{\delta\mathcal{J}_I}W[\,\mathcal{J},\gamma,\zeta\,], \quad \mathcal{J}_I = -\frac{\delta}{\delta\langle\varPhi^I\rangle}\varGamma[\,\langle\varPhi\rangle,\gamma,\zeta\,], \tag{13.52}$$

while the γ and ζ-derivatives of both functionals coincide,

$$\frac{\delta}{\delta\gamma}W[\,\mathcal{J},\gamma,\zeta\,]\Big|_{\mathcal{J}=\mathcal{J}[\,\langle\varPhi\rangle,\gamma,\zeta\,]} = \frac{\delta}{\delta\gamma}\varGamma[\,\langle\varPhi\rangle,\gamma,\zeta\,], \tag{13.53}$$

$$\frac{\delta}{\delta\zeta}W[\,\mathcal{J},\gamma,\zeta\,]\Big|_{\mathcal{J}=\mathcal{J}[\,\langle\varPhi\rangle,\gamma,\zeta\,]} = \frac{\delta}{\delta\zeta}\varGamma[\,\langle\varPhi\rangle,\gamma,\zeta\,]\,. \tag{13.54}$$

Problem 13.5. Prove that Eqs.(13.53)-(13.54) follow from (13.51)-(13.52)

Let us now use the relations (13.52)-(13.54) in Ward identities (13.48) and b-equations of motion (13.50). For brevity from now on we will discard the averaging symbol in the notation for mean fields, $\langle\,\varPhi\,\rangle \to \varPhi$, $\langle\, g^a\rangle \to g^a$, etc. Then we get for $\varGamma = \varGamma[\,g,C,\bar C,b,\gamma,\zeta\,]$ the following two equations

$$\frac{\delta\varGamma}{\delta\gamma_a}\frac{\delta\varGamma}{\delta g^a} + \frac{\delta\varGamma}{\delta\zeta_\alpha}\frac{\delta\varGamma}{\delta C^\alpha} + b_\alpha\frac{\delta\varGamma}{\delta\bar C_\alpha} = 0\;, \tag{13.55}$$

$$\chi^\alpha_a\varphi^a - c^{\alpha\beta}b_\beta - \frac{\delta\varGamma}{\delta b_\alpha} = 0\;. \tag{13.56}$$

Integration of the second of these equations immediately gives

$$\varGamma = \hat\varGamma + b_\alpha\big(\chi^\alpha - \frac{1}{2}c^{\alpha\beta}b_\beta\,\big), \tag{13.57}$$

where $\hat\varGamma = \varGamma[\,g,C,\bar C,\gamma,\zeta\,]$ is independent of the Lagrange multiplier b_α *reduced* action

$$\frac{\delta\hat\varGamma}{\delta b_\alpha} = 0. \tag{13.58}$$

Equation (13.57) actually implies disentangling from the full effective action the gauge breaking part written in terms of the gauge conditions functions χ^α and their Lagrange multiplier b_α (cf. this part of Eq.(13.28)).

Then, substitution of (13.57) into (13.55) shows that it splits into two equations for $\hat{\varGamma}$. One of them,

$$\chi^\alpha_a \frac{\delta\hat{\varGamma}}{\delta\gamma_a} + \frac{\delta\hat{\varGamma}}{\delta\bar{C}_\alpha} = 0\,, \tag{13.59}$$

shows that $\hat{\varGamma}$ depends on the mean ghost field $\bar{C}_\alpha$ in a special combination $\hat{\gamma}_a = \gamma_a - \bar{C}_\alpha\,\chi^\alpha_a$ with extra source γ_a,

$$\hat{\varGamma} = \hat{\varGamma}[\,g, C, \hat{\gamma}, \zeta\,], \quad \hat{\gamma}_a = \gamma_a - \bar{C}_\alpha\,\chi^\alpha_a. \tag{13.60}$$

Another equation finally leads to the long-sought *Zinn-Justin equation* for reduced action in the minimal sector

$$\frac{\delta\hat{\varGamma}}{\delta\hat{\gamma}_a}\frac{\delta\hat{\varGamma}}{\delta g^a} + \frac{\delta\hat{\varGamma}}{\delta\zeta_\alpha}\frac{\delta\hat{\varGamma}}{\delta C^\alpha} = 0. \tag{13.61}$$

Problem 13.6. Check the derivation of (13.59)-(13.61) from (13.55). Derive from (13.28) the expression for the tree-level ("classical" or prior to functional integration) reduced action $\hat{\varGamma}$ by subtracting the gauge fixing part and adding the source terms for the BRST transformations of g^a and C^α.

This equation is very important in renormalization theory of gauge models because it allows one to establish the BRST structure of UV counterterms and renormalized action of the theory. As one can see either from Eqs. (13.27) and (13.34) or from Eqs.(13.42) and (13.43), the BRST structure of the action in the path integral is rather universal. It is the sum of the classical gauge-invariant action $S[\,g\,]$ – the functional of the single original field – and the BRST exact term in the form of the nilpotent BRST-operator acting on the gauge fermion which is the object containing all gauge-fixing information.

However, actual calculation of the path integral leads to UV divergences which can be cancelled by local counterterms. This is guaranteed by the BPHZ theory, but these counterterms are some general local functionals of all fields – original gauge fields g, ghosts $C, \bar{C}$, and the Lagrange multipliers b, and a priori there is no guarantee that these counterterms will have have the same BRST structure. If not, then this structure gets broken, BRST symmetry violated together with the original gauge invariance. The theory becomes inconsistent, because the physical results for gauge-invariant observables start depending on arbitrary element of the calculational procedure – choice of gauge conditions and gauge-fixing matrix. The corner stone of the original construction – gauge independence of the path integral and on-shell scattering amplitudes – gets lost.

Fortunately, it turns out that Ward identities of the above type and Zinn-Justin equation for effective action confirm the preservation of the BRST structure also for renormalized theory. Physically meaningful and gauge invariant counterterms – local functionals of the original gauge field – renormalize the classical action $S[\,g\,]$ in Eqs.(13.42)-(13.43), other ghost-field dependent counterterms renormalize the gauge fermion but leave the structure of BRST-exact term intact. All this can be perturbatively attained in the form of the loop expansion in powers of $\hbar$ along with a special (generally nonlinear) renormalization of all quantum fields $\varPhi$. This renormalization (or infinite reparametrization) converts the renormalized integrand of the path integral to the BRST form of the above type. This can be done iteratively, loop by loop, within semiclassical expansion in $\hbar$ by finding the so-called *cohomologies* of the nilpotent BRST-operator. The cohomology method coming from pure math on supermanifolds – configuration spaces of boson-fermion variables – serves as an efficient tool of perturbative solution of Ward and Zinn-Justin equations for the UV divergent part of effective action. Modern and sufficiently self-contained presentation of this renormalization theory can be found in [*A. O. Barvinsky, D. Blas, M. Herrero-Valea, S. M. Sibiryakov, C. F. Steinwachs, Renormalization of gauge theories in the background-field approach, JHEP 07 (2018) 035, arXiv:1705.03480.*].

The above method of BRST-symmetry and Ward identities has led us to the structures which actually go beyond the class of gauge theories restricted by several simplicity assumptions – the closure of the gauge algebra of generators, their linearity in gauge fields and their linear independence (irreducibility), etc. These assumptions can be removed within BFV and BV framework mentioned above – the formalism based on the introduction of *anti-bracket* acting on the space of fields and *anti-fields*. This formalism allows to go beyond the main limitation of the Faddeev-Popov method – its restriction to closed generator algebras. Open algebras, in addition to the linear combinations of generators in the right-hand side of (11.46) with no longer constant *structure functions*, also have the terms proportional to the classical extremal $\delta S/\delta g^a$. For these algebras the path integral becomes more complicated than the Faddeev-Popov one. The total action becomes of higher power than two in ghost fields, and the BV technique (or BFV technique within Hamiltonian formalism) suggests a systematic way to find the coefficients of this ghost polynomial – the higher-order structure functions of the gauge algebra.

Interestingly, the anti-bracket structure of the BV method actually appears already in the Zinn-Justin equation with extra sources γ_a and ζ_α for BRST-transformations of gauge and ghost fields. Therefore we finish this lecture course by a very brief comment about it. This comment looks useful, in particular, because the equations generalizing the notions of BRST-operator $\boldsymbol{s}$ and gauge fermion Ψ to the extended ones $\boldsymbol{Q}$ and $\boldsymbol{\Psi}$ in (13.39)-(13.40) do not look systematic, while the anti-bracket formalism puts it in an orderly fashion. In this formalism the configuration space of fields Φ is supplemented by the set of *anti-fields* Φ^*_I of opposite statistics, $\epsilon(\Phi^*_I)=\epsilon(\Phi^I)+1$, and for any two functions A and B on the space of Φ and Φ^*, the anti-bracket is defined as

$$(A,B)=A\,\frac{\overleftarrow{\delta}}{\delta\Phi^I}\,\frac{\overrightarrow{\delta}}{\delta\Phi^*_I}\,B-A\,\frac{\overleftarrow{\delta}}{\delta\Phi^*_I}\,\frac{\overrightarrow{\delta}}{\delta\Phi^I}\,B\,. \tag{13.62}$$

In fact, this is a modification of the super-Poisson bracket (9.27) in which, however, Φ^* is conjugated to Φ not in the canonical sense, but in the sense of Grassmann parity, and the derivatives together with their contractions are the spacetime rather than the canonical ones,

$$\frac{\overleftarrow{\delta}}{\delta\Phi^I}\,\frac{\overrightarrow{\delta}}{\delta\Phi^*_I}\equiv\int dt\,\frac{\overleftarrow{\delta}}{\delta\Phi^I(t)}\,\frac{\overrightarrow{\delta}}{\delta\Phi^*_I(t)}. \tag{13.63}$$

In the above BRST construction the anti-fields Φ^*_I are just the sources for BRST-transformations of the fields Φ^I,

$$\Phi^I=g^a,C^\alpha,\bar{C}_\alpha,b_\alpha, \tag{13.64}$$

$$\Phi^*_I=\gamma_a,\,\zeta_\alpha,\,y^\alpha, \tag{13.65}$$

(the anti-field for the Lagrange multiplier is absent because for theories with closed gauge algebras its BRST transformation is vanishing $\boldsymbol{s}b_\alpha=0$). In terms of these notations Eqs.(13.39)-(13.40) for extended BRST operator and gauge fermion take a readable form

$$\boldsymbol{Q}=\boldsymbol{s}-(-1)^{\epsilon_I}\mathcal{J}_I\frac{\delta}{\delta\Phi^*_I},\quad \boldsymbol{\Psi}=\Psi(\Phi)-(-1)^{\epsilon_I}\Phi^*_I\Phi^I, \tag{13.66}$$

whereas Zinn-Justin equation becomes after the transition to the *minimal sector*, $\hat{\varGamma}=\hat{\varGamma}[\Phi_{\rm min},\Phi^*_{\rm min}]$, $\Phi_{\rm min}=(g^a,C^\alpha)$, $\Phi^*_{\rm min}=(\hat{\gamma}_a,\zeta_\alpha)$,

$$(\hat{\varGamma},\hat{\varGamma})_{\rm min}=0. \tag{13.67}$$

This is a famous *master equation* of BV formalism which reveals the sequence of structure functions of gauge algebra. These structure functions for closed algebras are exhausted by $R^a_\alpha, C^\alpha_{\beta\gamma}$ and form the basis for Feynman-DeWitt-Faddeev-Popov path integral, while in theories with open algebras they extend to higher orders and become coefficients of higher-order powers of ghost fields in the full action of the theory.