One-loop corrections to the E-type α-attractor models of inflation and primordial black hole production

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Primordial black holes (PBHs) [Zel'dovich & Novikov, 1967; Hawking, 1971]

PBHs may

- contribute to the dark matter in our universe [Carr, 1975; Chapline, 1975]
- generate the large-scale structure [Meszaros, 1975]
- provide seeds for the supermassive black holes [Carr & Rees, 1984]
- explain future gravitational wave observations [Acquaviva et al., 2004]

Starting from the 1980's, many PBHs formation mechanisms were proposed [Khlopov & Polnarev, 1980], [Berezin, Kuzmin, Tkachev, 1983], [Hawking, 1989], [Dolvgov & Silk, 1993].
In the most studied scenario PBHs come from large fluctuations generated by inflationary dynamics [Ivanov, Naselsky, Novikov, 1994].

Inflationary dynamics

The action for canonical single-field models of inflation reads

$$\mathcal{S} = \int \mathrm{d}^4 x \sqrt{-g} \left(\frac{1}{2} R - \frac{1}{2} \left(\partial_\mu \phi \right)^2 - V(\phi) \right) \ , \quad c = \hbar = M_{\mathrm{Pl}} = 1 \ , \tag{1}$$

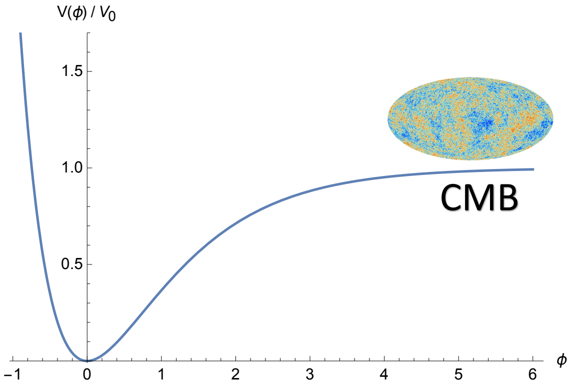
where ϕ is the inflaton field, and $V(\phi)$ is its potential. The Hubble rate, $H \equiv \dot{a}/a$, and ϕ satisfy the equations

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi = 0 \ , \quad 3H^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi) \ , \tag{2}$$

and the first and the second Hubble flow parameters are defined by

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{1}{2}\frac{\dot{\phi}^2}{H^2} \ , \quad \eta \equiv \frac{\dot{\epsilon}}{H\epsilon} \ . \tag{3}$$

Inflationary dynamics I



slow-roll

$$|\ddot{\phi}| \ll 3H|\dot{\phi}|$$

$$3H\dot{\phi} + V_\phi \simeq 0$$

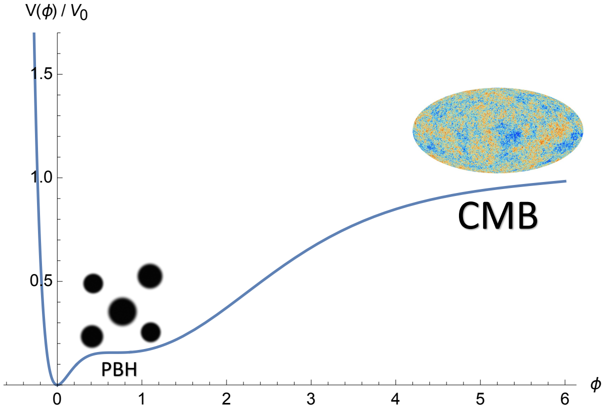
$$3H^2 \simeq V$$

$$\epsilon, |\eta| \ll 1$$

Inflationary dynamics II

ultra-slow-roll

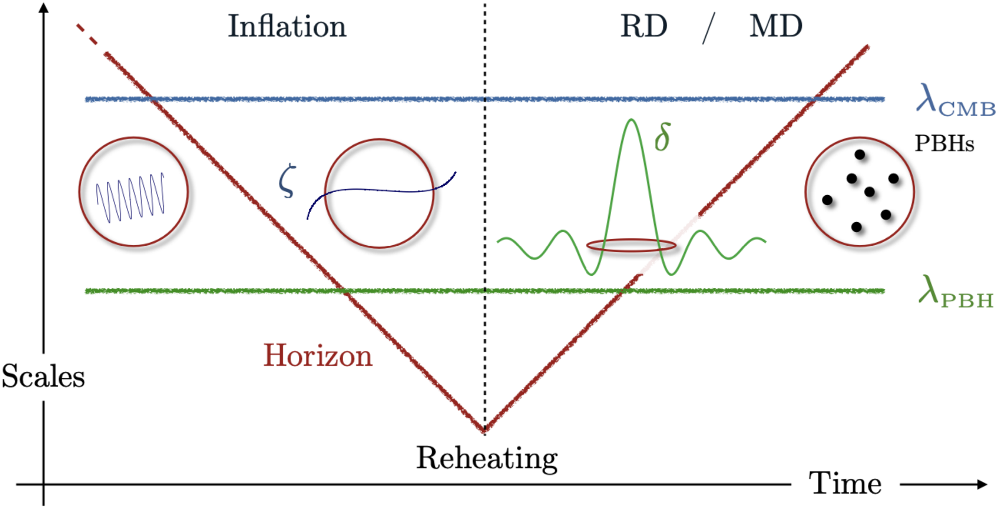
$$V_\phi \simeq 0$$
$$\ddot{\phi} + 3H\dot{\phi} \simeq 0$$
$$3H^2 \simeq V$$
$$\dot{\phi} \propto a^{-3}$$
$$\epsilon \propto a^{-6}\,,\ \eta \simeq -6$$



slow-roll

$$|\ddot{\phi}| \ll 3H|\dot{\phi}|$$
$$3H\dot{\phi} + V_\phi \simeq 0$$
$$3H^2 \simeq V$$
$$\epsilon, |\eta| \ll 1$$

The curvature perturbation in the comoving gauge, $\zeta_k(t)$, can be expressed in terms of the Mukhanov–Sasaki variable, v_k, as $\zeta_k \equiv v_k/z$, where $z \equiv a\dot{\phi}/H$. Thus, $\zeta_k \propto a^3$.



[Franciolini, 2021]

Our model

The basic α-attractor models are divided into two types depending upon the global shape of the inflaton scalar potential,

$$\text{E-type: } V \sim \left(1 - \exp\left(-\sqrt{\frac{2}{3\alpha}}\frac{\phi}{M_{\rm Pl}}\right)\right)^2, \quad \text{and} \quad \text{T-type: } V \sim \tanh^2 \frac{\phi/M_{\rm Pl}}{\sqrt{6\alpha}}\,. \tag{4}$$

The E-type potential can be modified to include PBH production [DF, Ketov, 2023] as

$$V(\phi) = \frac{3}{4}M^2 M_{\rm Pl}^2 \left[1 - y - \theta y^{-2} + y^2(\beta - \gamma y)\right]^2, \quad y = \exp\left(-\sqrt{\frac{2}{3\alpha}}\phi/M_{\rm Pl}\right). \tag{5}$$

This potential can be rewritten in terms of the new dimensionless parameters

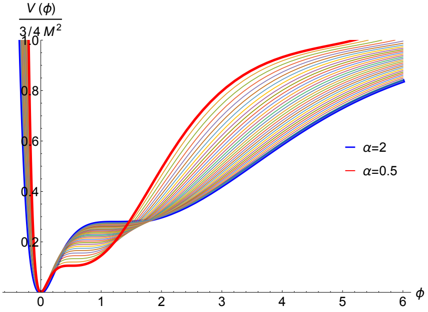
$$\beta = \frac{\exp\left(\sqrt{\frac{2}{3\alpha}}\phi_i\right)}{1-\sigma^2} \quad \text{and} \quad \gamma = \frac{\exp\left(2\sqrt{\frac{2}{3\alpha}}\phi_i\right)}{3(1-\sigma^2)}\,. \tag{6}$$

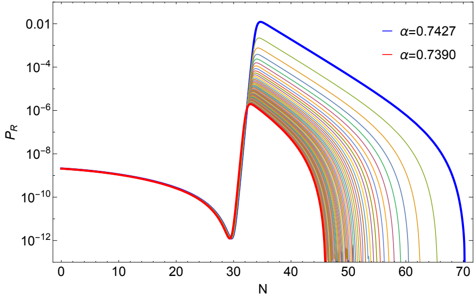
Two extrema are symmetrically located around the inflection point y_i as $y^{\pm}_{\rm ext} = y_i\,(1 \pm \sigma)$.

The power spectrum of scalar perturbations in the slow-roll approximation is

$$\mathcal{P}_R = \frac{H^2}{8\pi^2\epsilon} \; . \tag{7}$$

It is of primary interest when calculating the observable predictions of an inflation model.





One-loop corrections (1LC)

The primordial power spectrum is related to the 2-point correlator of curvature perturbations in Fourier space,

$$\langle \zeta_{\mathbf{k}} \zeta_{\mathbf{k}'} \rangle = (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') P_\zeta(k) \ , \quad \mathcal{P}_\zeta(k) = \frac{k^3}{2\pi^2} P_\zeta(k) \ . \tag{8}$$

Contributions to $\mathcal{P}_\zeta(k)$ can be organized into the loop-expansion, where the n^{th}-loop term contains n unconstrained momentum integrations,

$$\mathcal{P}_\zeta(k) = \mathcal{P}_\zeta(k)_{\text{tree}} + \mathcal{P}_\zeta(k)_{\text{1-loop}} + \ldots + \mathcal{P}_\zeta(k)_{\text{n-loop}} \ . \tag{9}$$

- [Kristiano, Yokoyama, Constraining PBH formation from single-field inflation, 2022]
- [Riotto, The PHB formation from single-field inflation is not ruled out, 2023]
- [Kristiano, Yokoyama, Note on the bispectrum and 1LC in single-field inflation with PBH formation, 2024]
- [Riotto, The PBH formation from single-field inflation is still not ruled out, 2024]

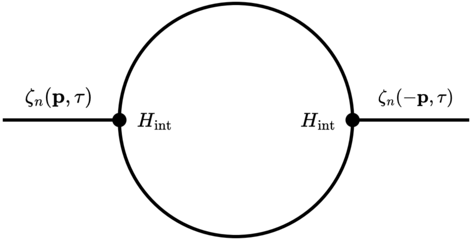
[and many more].

In the in-in formalism, the expectation value of an operator $\mathcal{O}$ at time τ is

$$\langle \mathcal{O}(\tau) \rangle = \left\langle \left[\bar{T} \exp \left(i \int_{-\infty(1+i\epsilon)}^{\tau} \mathrm{d}\tau' H_{\rm int}(\tau') \right) \right] \mathcal{O}(\tau) \left[T \exp \left(-i \int_{-\infty(1-i\epsilon)}^{\tau} \mathrm{d}\tau' H_{\rm int}(\tau') \right) \right] \right\rangle .$$

To compute 1LC, we set $\mathcal{O}(\tau) = \zeta(\mathbf{p}, \tau)\zeta(-\mathbf{p}, \tau)$, where $\mathbf{p}$ is the comoving wavenumber at large scale where we want to compute the correction,

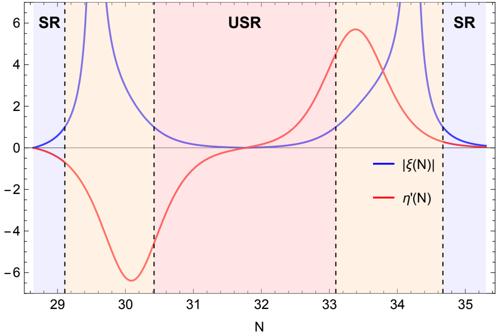
$$\langle \zeta(\mathbf{p},\, \tau)\zeta(-\mathbf{p},\, \tau) \rangle_{\text{1-loop}} = (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}) P_\zeta(p;\, \tau)_{\text{1-loop}} . \qquad (10)$$



$$H_{\text{int}} = -\frac{1}{2} M_{\rm pl}^2 \int d^3x \, \epsilon \, \eta' \, a^2 \, \zeta_n' \, \zeta_n^2$$

For a numerical calculation of 1LC to the scalar power spectrum, we adopted the result from [Davies, Iacconi, Mulryne, 2024] that takes into account cubic interactions,

$$\mathcal{P}_\zeta(p;N)_{\text{1-loop}} = \frac{p^3}{\pi^4}\int_{N_i}^{N_f} \mathrm{d}N_1\,\epsilon(N_1)\frac{\mathrm{d}\eta(N_1)}{\mathrm{d}N_1}\left(a(N_1)\right)^2 \int_{N_i}^{N_1} \mathrm{d}N_2\,\epsilon(N_2)\frac{\mathrm{d}\eta(N_2)}{\mathrm{d}N_2}\left(a(N_2)\right)^2$$
$$\int_{k_s}^{k_e} \mathrm{d}k\,k^2\,\mathrm{Im}\Bigg[\zeta_p(N)\,a(N_1)\,H(N_1)\,a(N_2)\,H(N_2)\,\frac{\mathrm{d}}{\mathrm{d}N_2}\left[\zeta_p^*(N_2)\left(\zeta_k^*(N_2)\right)^2\right] \tag{11}$$
$$\left\{\zeta_p^I(N)\frac{\mathrm{d}}{\mathrm{d}N_1}\left[\zeta_p^R(N_1)\left(\zeta_k(N_1)\right)^2\right] - \zeta_p^R(N)\,\frac{\mathrm{d}}{\mathrm{d}N_1}\left[\zeta_p^I(N_1)\left(\zeta_k(N_1)\right)^2\right]\right\}\Bigg]\,.$$



The standard Hubble-flow parameters are

$$\epsilon_{i+1} = \epsilon_i'/\epsilon_i\ , \quad \epsilon_0 = H^{-1}\ . \tag{12}$$

To distinguish the different phases of inflation and compute 1LC, we used

$$\epsilon \equiv \epsilon_1\ , \quad \eta \equiv \epsilon_2\ , \quad \xi \equiv \epsilon_3\ . \tag{13}$$

To determine the transition periods, we used the condition $|\xi| \geq 1$.

Reconstruction of scalar potential from GW signal

GW signal $\Omega_{\rm GW}(k)$ → **Power spectrum** $\mathcal{P}_\zeta(k)$ → **Scalar potential** $V(\phi)$

The present GW energy density computed in the second order is given by

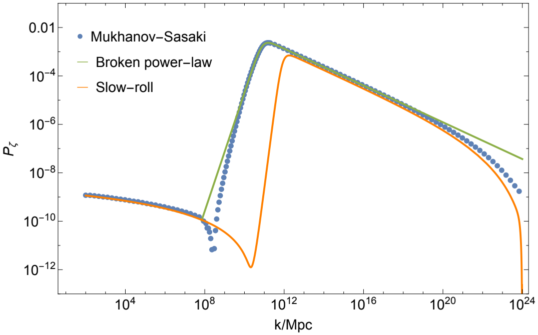
$$\frac{\Omega_{\rm GW}(k)}{\Omega_r} = \frac{c_g}{72} \int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \mathrm{d}\, d \int_{\frac{1}{\sqrt{3}}}^{\infty} \mathrm{d}\, s \left[\frac{\left(s^2 - \frac{1}{3}\right)\left(d^2 - \frac{1}{3}\right)}{s^2 + d^2} \right]^2 \mathcal{P}_\zeta(kx)\, \mathcal{P}_\zeta(ky)\, (I_c^2 + I_s^2)\,, \qquad (14)$$

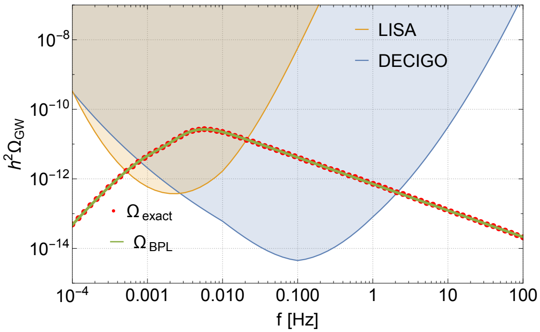
where $x = \frac{\sqrt{3}}{2}(s + d)$, and $y = \frac{\sqrt{3}}{2}(s - d)$, and the functions I_c and I_s are given by

$$I_s = -36 \frac{s^2 + d^2 - 2}{(s^2 - d^2)^2} \left[\frac{s^2 + d^2 - 2}{s^2 - d^2} \log\left| \frac{d^2 - 1}{s^2 - 1} \right| + 2 \right], \quad I_c = -36\pi \frac{\left(s^2 + d^2 - 2\right)^2}{\left(s^2 - d^2\right)^3} \theta(s-1)\,,$$

see [Espinosa, Racco, Riotto, 2018], and [Kohri & Terada, 2018].

The procedure of reconstruction is ambiguous. Nevertheless, it is possible to systematically identify the power spectra corresponding to a specific GW energy density curve [DF, Kühnel, Stamou, 2024], [LISA Cosmology Working Group, 2025].





$$\mathcal{P}_{\rm BPL} = A\frac{\alpha_1+\beta_1}{\beta_1(k/k_*)^{-\alpha_1}+\alpha_1(k/k_*)^{\beta_1}}\,, \qquad S(\vec{\theta}) = \sum_i \left[\mathcal{P}_\zeta(k_i;\vec{\theta}) - \mathcal{P}_\zeta^{\rm rec}(k_i)\right]^2.$$

Our results

The key observables and the relative values of the 1LC in our model:

E-Model	n_s	r	h	$\Delta N_{\rm USR}$	$M_{\rm PBH}$, g	$\mathcal{P}_\zeta(k_{\rm peak})$	$\delta_{\rm 1L}$, %
Set 1	0.9649	0.01466	-1.472	2.674	$2.2 \cdot 10^{21}$	10^{-3}	1.31
Set 2	0.9649	0.01685	-1.435	2.838	$7.1 \cdot 10^{22}$	$0.5 \cdot 10^{-2}$	2.96
Set 3	0.9649	0.014323	-1.471	2.687	$4.6 \cdot 10^{18}$	10^{-3}	1.33

The CMB tilts n_s and r, the sharpness parameter h, the USR phase duration $\Delta N_{\rm USR}$, the PBH masses $M_{\rm PBH}$, the peak amplitude $\mathcal{P}_\zeta(k_{\rm peak})$, and the relative 1LC $\delta_{\rm 1L}$ for the different parameter sets

- The 1LC in our E-model does not exceed a few percent, being primarily depending upon the amplitude of the peak in the scalar power spectrum
- The inverse reconstruction of the parameters in the E-model demonstrates its potential as a reasonable framework for interpreting future GW observations

The E-model of inflation and PBH production is constrained (fine-tuned) but not ruled out by CMB measurements and one-loop corrections

Thank you for your attention!