Entanglement islands in geometries with horizons

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Definition of entanglement entropy

- Assume that the system is in the ground state $|0\rangle \implies \rho_{\text{tot}} = |0\rangle\langle 0|$.
- The total Hilbert space is assumed to have a product form according to some partition of dofs of the system $X \cup \overline{X}$:

$$\mathcal{H}_{\text{tot}} \simeq \mathcal{H}_X \otimes \mathcal{H}_{\overline{X}}. \tag{1}$$

- The reduced density matrix of a subsystem X is defined as

$$\rho_X = \operatorname{Tr}_{\mathcal{H}_{\overline{X}}} \left(\rho_{\text{tot}}\right). \tag{2}$$

- The entanglement entropy (EE) of a subsystem X is defined as the von Neumann entropy for the reduced density matrix ρ_X:

$$S_X = -\operatorname{Tr}_X \left(\rho_X \log \rho_X\right). \tag{3}$$

Basic properties of entanglement entropy

EE measures how much a given (generically entangled) pure state $|\psi_e\rangle$:

$$|\psi_e\rangle = \sum_{i\in X,\ j\in\overline{X}} c_{ij}\,|i\rangle_X \otimes |j\rangle_{\overline{X}}, \tag{4}$$

differs from a separable state $|\psi_s\rangle = |X\rangle \otimes |\overline{X}\rangle$ with $S_{X,\overline{X}} = 0$.

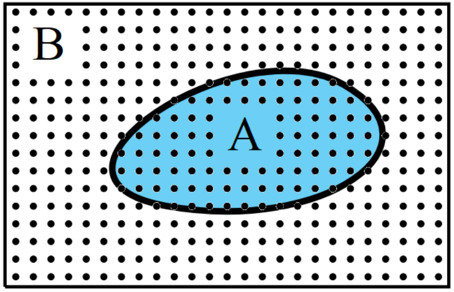
Claim: EE vanishes $\iff$ the pure ground state is separable.

Basic properties of EE:

- Purity: if the state ρ_{tot} is pure, then $\rho_{\text{tot}}^2 = \rho_{\text{tot}} \implies S_{\text{tot}} = 0$.
- Complementarity: if the state ρ_{tot} of the system $X \cup \overline{X}$ is pure, then $S_X = S_{\overline{X}}$.
- Araki-Lieb triangle inequality: $|S_X - S_{\overline{X}}| \leq S_{X\cup\overline{X}}$.

Quantum Mechanics vs. QFT

In QM, the trace in ρ_X is taken over the dofs of the subsystem $\overline{X}$ (e.g. spins).



In QFT, in vacuum state $|0\rangle$ there are no particles, and entanglement is calculated not between the dofs but the subregions (x is a spacelike coordinate): $\mathcal{H}_{\text{tot}} = \bigotimes_x \mathcal{H}_x, \ \mathcal{H} \simeq \mathcal{H}_A \otimes \mathcal{H}_B, \ \mathcal{H}_X = \bigotimes_{x \in X} \mathcal{H}_x, \ X = A, B.$
Associated decomposition of the lattice into subregions corresponds to factorization of the total Hilbert space $\mathcal{H}_{\text{tot}}$.

Reniy entropy and replica trick

- In a general QFT, the calculation of EE is an extremelly challenging problem.
- The Renyi entropy $S_n(X)$ is a one-parameter (n — replica parameter) generalization of EE:

$$S_n(X) = -\frac{1}{n-1} \log \operatorname{Tr}_X \left(\rho_X^n\right), \quad n \in \mathbf{N}. \tag{5}$$

  Analytic continuation $n \in \mathbf{N} \to n \in \mathbf{R}$ allows to compute EE in QFT.
- Replica trick: EE S_X can be calculated as the $n \to 1$ limit ($n \in \mathbf{R}$) of $S_n(X)$ with the normalization $\operatorname{Tr}_X (\rho_X) = 1$:

$$S_X = \lim_{n\to 1} S_n(X) = -\lim_{n\to 1} \partial_n \log \operatorname{Tr}_X \left(\rho_X^n\right). \tag{6}$$

CFT_2

- CFT_2 is a $d = 2$ QFT invariant under the transformations of the underlying metric (I_E — Euclidean action):

$$x \to \overline{x}, \tag{7}$$

$$\overline{g}_{\mu\nu}(\overline{x}) = \Omega^2(x) g_{\mu\nu}(x), \tag{8}$$

$$I_E\left[g_{\mu\nu}, \phi\right] = I_E\left[\overline{g}_{\mu\nu}, \overline{\phi}\right]. \tag{9}$$

- In CFT_2, we do not need to specify the action I_E in order to define correlation functions.
- Primary operator in CFT is a local operator $O(x)$ with the following transformation property:

$$\overline{O}\left(\overline{x}\right) = \Omega^{-\Delta}(x) O(x), \quad x \to \overline{x}. \tag{10}$$

CFT_2

- In CFT_2, in the vacuum state on a plane, the lowest correlation functions for primary operators $O_{h\bar{h}}(z,\bar{z})$ with conformal weights $h, \bar{h}$ $(h+\bar{h}=\Delta)$ are fully fixed by conformal symmetry:

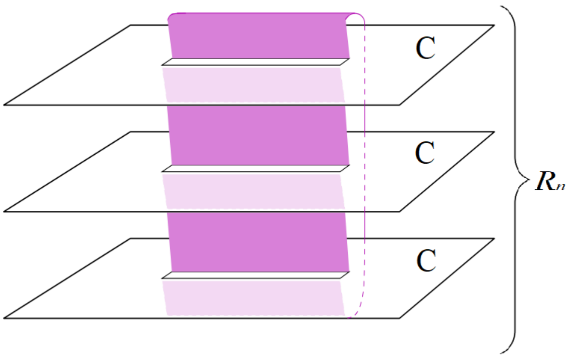
$$\langle O_{h\bar{h}}(z,\bar{z})\rangle = 0, \tag{11}$$

$$\langle O_{h\bar{h}}(z_1,\bar{z}_1) O_{h\bar{h}}(z_2,\bar{z}_2)\rangle = \frac{C}{(z_1-z_2)^{2h}(\bar{z}_1-\bar{z}_2)^{2\bar{h}}}. \tag{12}$$

- Transformation law of correlation functions for primary operators $O_i(z_i,\bar{z}_i)$ under $z \to z(w)$, $\bar{z} \to \bar{z}(\bar{w})$:

$$\langle O_1(w_1,\bar{w}_1)...O_1(w_n,\bar{w}_n)\rangle = \prod_{i=1}^{n} \left(\frac{dw}{dz}\right)^{-h_i}_{w=w_i} ... \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}_i}_{\bar{w}=\bar{w}_i} \langle O_1(z_1,\bar{z}_1)...O_1(z_n,\bar{z}_n)\rangle. \tag{13}$$

Replica trick in CFT_2



Let A be an interval (a, b) on a line $\mathbf{R}$.
In CFT_2, due to infinite-dimensional symmetry, it can be shown that the vacuum state ρ_A^n on the replica manifold $\mathcal{R}_n$ and the state ρ on a plane $\mathbf{C}$ created by the insertion at the endpoints a and b of the so-called twist operators $O_{\Delta_n}(z, \bar{z})$, which are primaries of special type with the conformal dimension:

$$\Delta_n = \overline{\Delta}_n = \frac{c}{24}\left(1 - \frac{1}{n^2}\right), \tag{14}$$

are equivalent:

$$\langle \mathcal{O}_A \rangle_{\mathcal{R}_n} = \operatorname{Tr}\left(\mathcal{O}_A \rho_A^n\right)_{\mathcal{R}_n} = \operatorname{Tr}\left(\mathcal{O}_A \rho\right)_{\mathbf{C}} = \tag{15}$$

$$= \langle \mathcal{O}_A O_{\Delta_n}(a) O_{\Delta_n}(b) \rangle_{\mathbf{C}}. \tag{16}$$

Replica trick in CFT_2

- This allows to compute EE of an interval A of length ℓ in CFT_2 [*Cardy & Calabrese'04*]:

$$S_A = -\frac{1}{n-1} \lim_{n\to 1} \mathrm{Tr}\,(\rho_A^n) = \frac{c}{3}\log\frac{\ell}{\varepsilon}, \tag{17}$$

  where ε is a UV cutoff.
- With this result and due to conformal symmetry in $d = 2$, EE can be calculated in any geometry conformally related to $\mathbf{C}$.
- EE of N disjoint intervals $[x_1, y_1] \cup \ldots \cup [x_N, y_N]$ in CFT_2 with free massless Dirac fermions is given by [*Casini et al.'05*]:

$$S = \frac{c}{3}\left(\sum_{i,j=1}^{N} \log\frac{|x_i - y_j|}{\varepsilon} - \sum_{i<j}^{N} \log\frac{|x_i - x_j|}{\varepsilon} - \sum_{i<j}^{N} \log\frac{|y_i - y_j|}{\varepsilon}\right). \tag{18}$$

BIG PICTURE MOTIVATION

- Due to semiclassical effects (quantum matter fields + classical gravitational field) black holes emit approximately thermal radiation and possibly evaporate completely [*Hawking'74*].
- The information paradox for black holes is an unresolved fundamental problem of modern "quantum" (semiclassical) gravity [*Hawking'76*].
- Can the island approach [*Almheiri et al.'19,'20; Penington'20*] be the solution to the information paradox?
- Our universe is currently approaching de Sitter spacetime, which is also a geometry with an event horizon.
- Cosmological and black hole horizons share similar properties: temperature, entropy, Hawking radiation [*Gibbons, Hawking'77*].
- However, cosmological horizon is observer-dependent, in contrast to black hole geometry. What is the microscopic interpretation of the cosmological horizon?

INFORMATION PARADOX

- The information paradox can be formulated in terms of the time dependence of EE of Hawking radiation.
- The Hilbert space of the system "black hole (BH) + outside radiation (R)" is given by:

$$\mathcal{H}_{\rm tot} = \mathcal{H}_{BH} \otimes \mathcal{H}_R, \quad \rho_{tot}^2 = \rho_{tot}. \tag{19}$$

- Let a black hole be formed from the collapse of matter in a pure state. Then we have [*Page'93,'13; Bekenstein'73*]:

$$\rho_{\rm tot}^2 = \rho_{\rm tot} \Rightarrow S(R) = S(BH) \leq S^{thermod}(BH) \propto \text{Area(horizon)}. \tag{20}$$

Information paradox

- When the black hole evaporates completely, there is only thermal radiation. Thus, the evolution of the state ρ_{tot} of a closed system $BH \cup R$ is given by:

$$\rho_{\text{tot,f}} = U(t)\rho_{\text{tot,0}}U^\dagger(t), \quad U^\dagger(t)U(t) = U(t)U^\dagger(t) = 1. \tag{21}$$

- Therefore, unitary evolution implies $\rho^2_{\text{tot}}(t) = \rho_{\text{tot}}(t)\ \forall t$.
- At the same time:

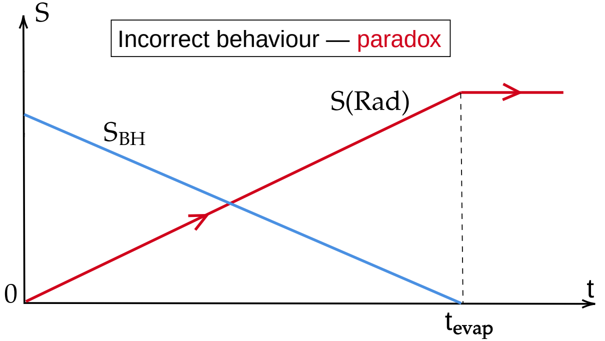
$$\rho^2_{\text{tot,0}} = \rho_{\text{tot,0}} \quad \text{initial pure state,} \tag{22}$$

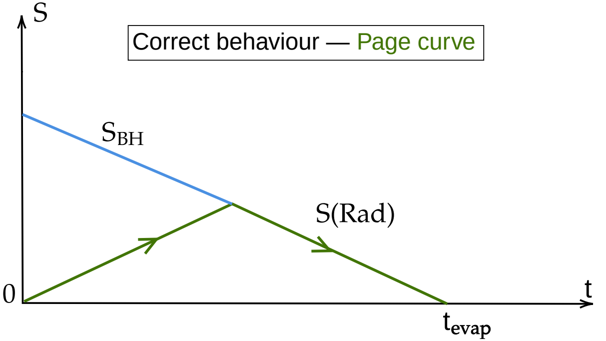
$$\rho^2_{\text{tot,f}} \neq \rho_{\text{tot,f}} \quad \text{final mixed (thermal) state.} \tag{23}$$

Under time evolution of ρ_{tot} we have: pure state $\rightarrow$ mixed state. This violates unitarity of quantum mechanics $\implies$ infotmation paradox (non-unitary evolution of Hawking radiation).

INFORMATION PARADOX

- Two possible ways to adress the problem: either to give up unitarity of quantum gravity or to find the solution to the paradox.
- Holographic duality [*Maldacena'97; Witten'98*] strongly suggests that the theory of the full quantum gravity must be unitary. Therefore, the information paradox must be resolved in the framework of quantum mechanics.





Entropy in the presence of gravity

The island rule for EE of QFT in systems with dynamical gravity reads [Almheiri et al.'19,'20; Penington'20; Penington et al.'22]:

$$S(R) \simeq \min_{\mathcal{I}} \left\{ \operatorname*{ext}_{\mathcal{I}} \left[\frac{\text{Area}(\partial\mathcal{I})}{4G_N} + S_{\text{m}}(R \cup \mathcal{I}) \right] \right\} , \tag{24}$$

where

- $\Sigma \simeq R \cup \overline{R}$ is the Cauchy surface, on which the vacuum state ρ_{tot} is defined (Hartle-Hawking for black holes, Bunch-Davies for de Sitter);
- R is the entangling region;
- $\mathcal{I}$ is the entanglement island (quantum extremal surface [Ruy & Takayanagi'06; Hubeny et al.'07; Engelhardt & Wall'14]) defined by extremization of the generalized entropy functional (24), and $\partial\mathcal{I}$ is its boundary;
- S_{m} is EE of QFT on the fixed classical background.

Entanglement entropy in higher-dimensional setups

Calculation of EE in $d > 2$ curved spacetime is challenging. The main difficulties are as follows:

- the island rule (24) is derived only in holography and in JT gravity;
- S_{m} can be calculated only in some special cases. One of few such cases is the replica trick in CFT_2.

Thus, the problem should be reduced to CFT_2 somehow. Main approaches:

- s-wave approximation:

$$S_{\text{pure geometry}} + S_{\text{matter}} \xrightarrow{\text{s-wave}} S_{\mathrm{CFT}_2}, \tag{25}$$

- dimensional reduction of pure (matter-free) $d > 2$ geometry to $d = 2$ with consideration of CFT_2 on the resulting background:

$$S_{\text{pure geometry}} \xrightarrow{\text{dimensional reduction}} S_{\text{reduced geometry}} + S_{\mathrm{CFT}_2}. \tag{26}$$

S-WAVE APPROXIMATION

- The massless field $\varphi(x^\mu)$ of QFT on a spherically symmetric $d > 2$ background is decomposed into spherical harmonics Y_{lm}:
  $\varphi(r,t,\theta,\phi) = \sum_{l,m} Y_{lm}(\theta,\phi) f_l(r,t)$.
- After this expansion, we obtain a set of effective $d = 2$ massive scalar field theories with masses $m^2 \propto l(l+1)$.
- The lowest $l = 0$ harmonic (s-wave) corresponds to an effective massless $d = 2$ theory.
- Neglecting $l > 0$ harmonics due to the effective gravitational barrier-type potential around the horizon, we *assume* that s-mode corresponds to CFT_2 and that EE of such effective theory approximates EE of the original problem [*Penington'20; Hashimoto et al.'20*].

INFORMATION PARADOX FOR FINITE REGIONS

- Let us divide a Cauchy surface in a two-sided black hole geometry into the "black hole region" BH, a finite entangling region R and an adjacent semi-infinite region C, which extends to spacelike infinities i^0:

$$\Sigma = BH \cup R \cup C. \tag{27}$$

- Strong subadditivity of EE gives [*Solodukhin'11; Nishioka'18*]:

$$S(BH \cup R \cup C) + S(R) \leq S(BH \cup R) + S(R \cup C). \tag{28}$$

- Complementarity of EE:

$$\begin{aligned} S(BH) &= S(R \cup C), \\ S(R) &= S(BH \cup C), \\ S(C) &= S(BH \cup R). \end{aligned} \tag{29}$$

Information paradox for finite regions

- Then we obtain the strong bound on EE:

$$S(R) \leq 2S_{\rm BH} + S(C), \quad S_{\rm BH} = \frac{\text{Area (horizon)}}{4G}. \tag{30}$$

- The violation of this bound can be seen as the information paradox for finite entangling regions.
- The island proposal softens this constraint and gives *the soft bound*:

$$S(R) \leq 2S_{\rm BH} + S(C) + S_{\rm corr}. \tag{31}$$

The correction $S_{\rm corr}$ is time-independent and small compared to the area term $S_{\rm BH}$ under the "black hole classicality" condition: $r_h^2/G \gg c$ [*Hashimoto et al.'20*].

Schwarzschild black hole: geometry

The metric of $d = 4$ Schwarzschild black hole is:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2, \quad f(r) = 1 - \frac{r_h}{r}, \quad \kappa_h = \frac{1}{2r_h} = 2\pi T_H. \tag{32}$$

The radial geodesic distance $d(\mathbf{x}, \mathbf{y})$ between the points $\mathbf{x} = (t_x, x)$ and $\mathbf{y} = (t_y, y)$ is given by:

$$d^2(\mathbf{x}, \mathbf{y}) = \frac{2\sqrt{f(x)f(y)}}{\kappa_h^2}\left[\cosh \kappa_h(r_*(x) - r_*(y)) - \cosh \kappa_h(t_x - t_y)\right], \tag{33}$$

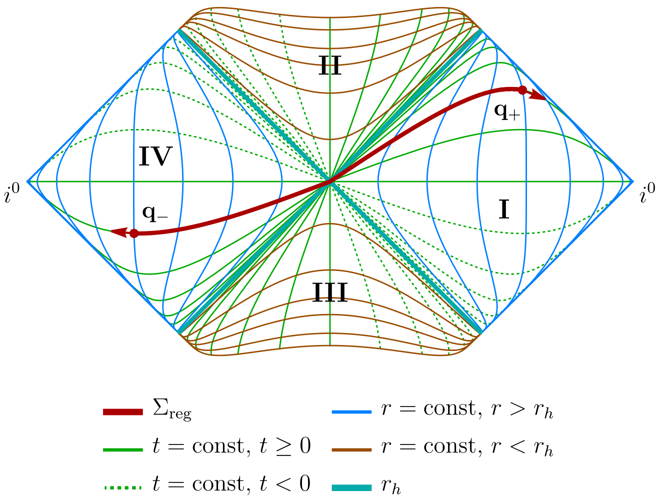
$$r_*(r) = r + r_h \log\left(\frac{r - r_h}{r_h}\right). \tag{34}$$

Given the spherical symmetry of this geometry, we use the s-wave approximation.

Schwarzschild black hole: goals

- Radiation is collected in the outer region R (generically infinite) with respect to the black hole horizon. To study $S(R)$, usually time evolution of the finite complement $\overline{R}$ is calculated, since the total state is pure: $S(R) = S(\overline{R})$.
- IR regularization of EE allows to directly calculate time evolution of $S(R)$ and explicitly check purity and complementarity in the described setup.
- IR regularization of EE allows to formulate the version of the information paradox for finite entangling regions.
- S-wave reduction allows to apply the island rule to higher-dimensional spherically-symmetric geometries.
- The setup with finite regions is also motivated by Schwarzschild-de Sitter spacetime, in which there is a natural bound on the size of entangling regions determined by the cosmological horizon.

SCHWARZSCHILD BLACK HOLE: IR REGULARIZATION



IR regularization of Cauchy surface Σ:

$$\mathbf{q}_+ = \Big(q_+,\ t_{q_+}(q_+)\Big), \tag{35}$$

$$\mathbf{q}_- = \left(q_-,\ t_{q_-}(q_-) + \frac{i\pi}{\kappa_h}\right). \tag{36}$$

EE of the Hartle-Hawking state defined on Σ is then given by

$$S_{\text{m}}\left(\Sigma_{\text{reg}}\right) = \frac{c}{3}\log\frac{d(\mathbf{q}_-,\ \mathbf{q}_+)}{\varepsilon} = \frac{c}{3}\log\frac{2}{\kappa_h\varepsilon}. \tag{37}$$

The same calculations can be made for $R \subset \Sigma$ and $\overline{R} = \Sigma / R$.

Schwarzschild black hole: IR regularization

Basic properties of EE of a pure total state are violated in the background of an eternal asymptotically flat spherically-symmetric black hole, namely:

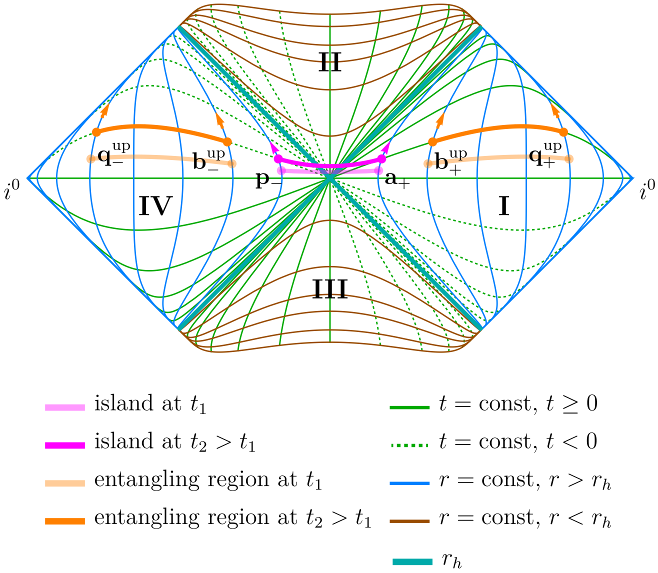
- $S(\Sigma) = \frac{c}{3} \log \frac{2}{\kappa_h \varepsilon} \neq 0.$
- $S\left(\overline{R}\right) - S(R) = \frac{c}{3} \log \frac{2}{\kappa_h \varepsilon} \neq 0$, where $R \subset \Sigma$ and $\overline{R} = \Sigma / R$.

This constant that violates the properties of EE:

- UV diverges at $\varepsilon \to 0$.
- can be arbitrarily large in the near-extremal ($GM^2 \to Q^2$) Reissner-Nordström black hole ($\kappa_h \to 0$).

We prescribe to renormalize all calculations by subtracting this constant.

Schwarzschild black hole: finite regions



- The region $R \equiv [\mathbf{q}_-, \mathbf{b}_-] \cup [\mathbf{b}_+, \mathbf{q}_+]$, where $\mathbf{b}_+ = (b, t_b)$, $\mathbf{q}_- = (b, -t_b + i\pi/\kappa_h)$, $\mathbf{q}_+ = (q, t_b)$, $\mathbf{q}_- = (q, -t_b + i\pi/\kappa_h)$ is a union of domains between two concentric spheres with radii b and $q > b$ in two-sided black hole geometry.
- At early times: $S_{\mathrm{m}}(R) \simeq \dfrac{2c}{3}\kappa_h t_b$.
- At late times: $S_{\mathrm{m}}(R) = \mathrm{const}$.
  This saturation of EE (without islands!) can be interpreted as an equilibrium of incoming and outgoing fluxes of Hawking radiation through the boundaries b and q.

SCHWARZSCHILD BLACK HOLE: ISLANDS OF FINITE LIFETIME

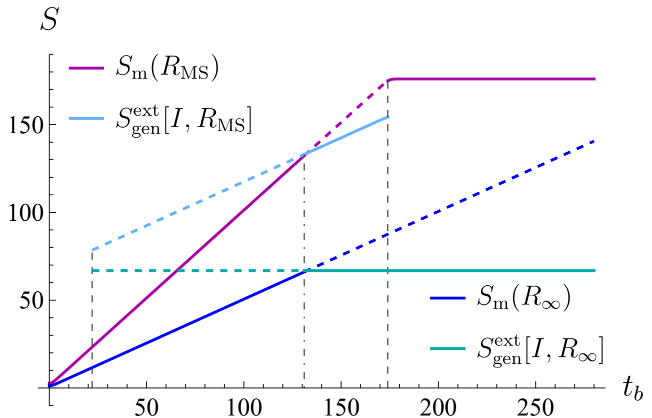


Figure. Evolution of $S(R)$ and $S(R_\infty)$ (solid lines). Non-dominating configurations are marked dashed. There is a discontinuity of EE of Hawking radiation for the finite-size configuration (dark magenta).

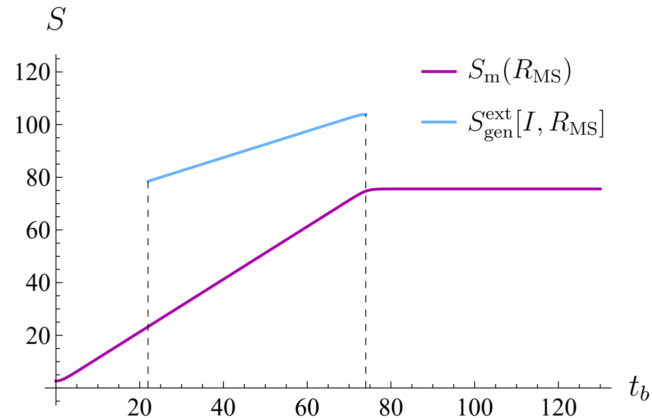
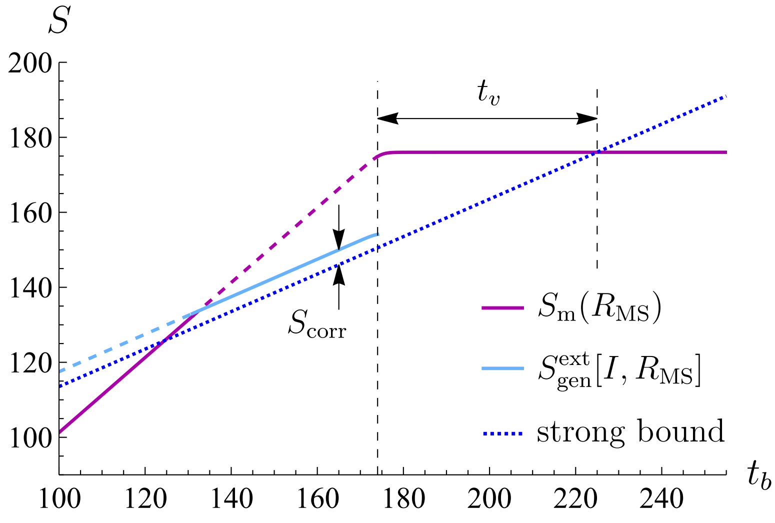


Figure. Evolution of $S_{\rm m}(R)$ and $S^{\rm ext}_{\rm gen}[I, R]$. The value of EE of Hawking radiation collected in R is given by the minimum of these two curves. During the entire finite lifetime, the island never dominates.

Schwarzschild black hole: information paradox for finite regions



Evolution of $S_{\rm m}(R)$, $S_{\rm gen}^{\rm ext}[I, R]$ and the strong bound (dotted blue) for a finite-size entangling region R. EE of Hawking radiation at each moment is given by the minimum of $S_{\rm m}(R)$ and $S_{\rm gen}^{\rm ext}[I, R]$ (solid curves). When the island disappears, there is a discontinuity. The larger the value of q is — the longer the time t_v of violation of the strong bound. Thus, the island prescription does not resolve the information paradox completely in the given setup.

Reissner-Nordström black hole: geometry

The metric of $d = 4$ Reissner-Nordström black hole is:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2, \quad f(r) = \frac{1}{r^2}(r - r_+)(r - r_-), \tag{38}$$

$$r_\pm \equiv GM \pm \sqrt{(GM)^2 - GQ^2}. \tag{39}$$

The Hawking temperature T_H and the surface gravity κ_h read:

$$T_H = \frac{r_+ - r_-}{4\pi r_+^2} = \frac{1}{2\pi} \frac{\sqrt{(GM)^2 - GQ^2}}{\left(GM + \sqrt{(GM)^2 - GQ^2}\right)^2}, \tag{40}$$

$$\kappa_h = 2\pi T_H = \frac{r_+ - r_-}{2r_+^2}. \tag{41}$$

Reissner-Nordström black hole: geometry

The radial geodesic distance $d(\mathbf{x}, \mathbf{y})$ between the points $\mathbf{x} = (t_x, x)$ and $\mathbf{y} = (t_y, y)$ is:

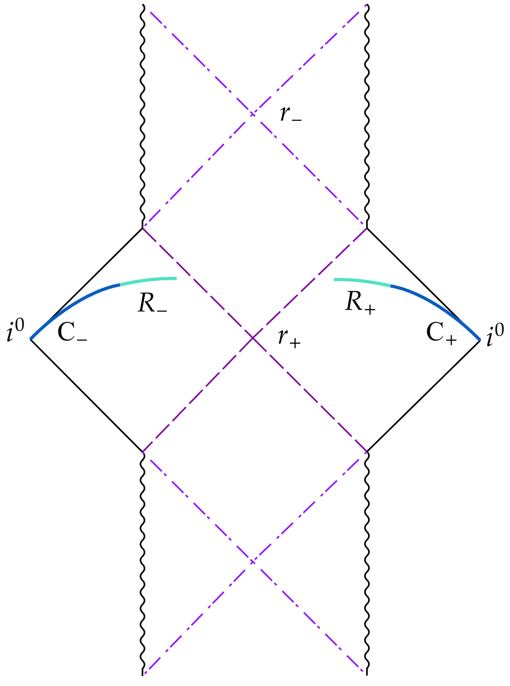
$$d^2(\mathbf{x}, \mathbf{y}) = \frac{2\sqrt{f(x)f(y)}}{\kappa_h^2}\left[\cosh \kappa_h(r_*(x) - r_*(y)) - \cosh \kappa_h(t_x - t_y)\right], \tag{42}$$

$$r_*(r) = r + \left(\frac{r_+^2}{r_- - r_+}\right) \ln\left|\frac{r - r_+}{r_+}\right| - \left(\frac{r_-^2}{r_+ - r_-}\right) \ln\left|\frac{r - r_-}{r_-}\right|. \tag{43}$$

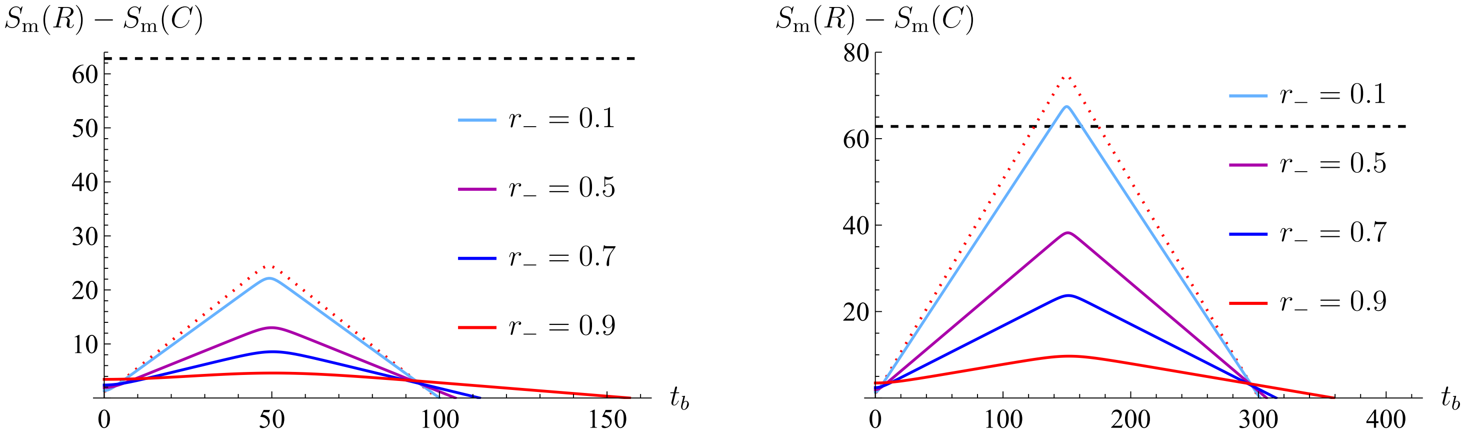
Near-extremal Reissner-Nordström black hole is defined as the charged black hole with $r_- \to r_+$ ($T_H \to 0$).

Given the spherical symmetry of this geometry, in calculations we also use the s-wave approximation.

Reissner-Nordström black hole: geometry

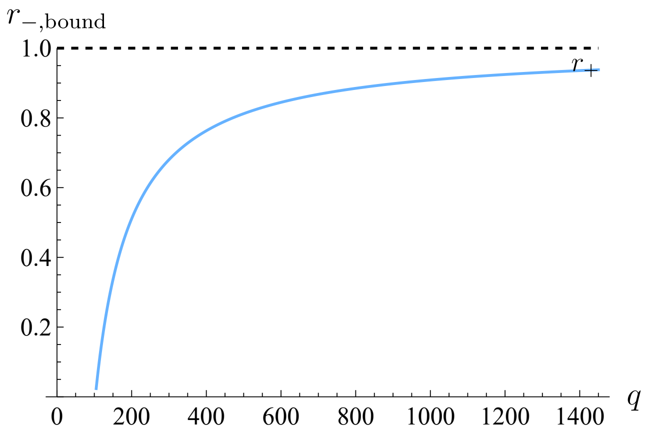


Reissner-Nordström black hole: strong bound



Evolution of $S_{\mathrm{m}}(R) - S_{\mathrm{m}}(C)$ in Reissner-Nordström black hole with different r_-. The red dotted curves denote the same for Schwarzschild black hole ($r_- = 0$), and the black dashed lines depict $2S_{\text{B-H}}$.

Reissner-Nordström black hole: strong bound



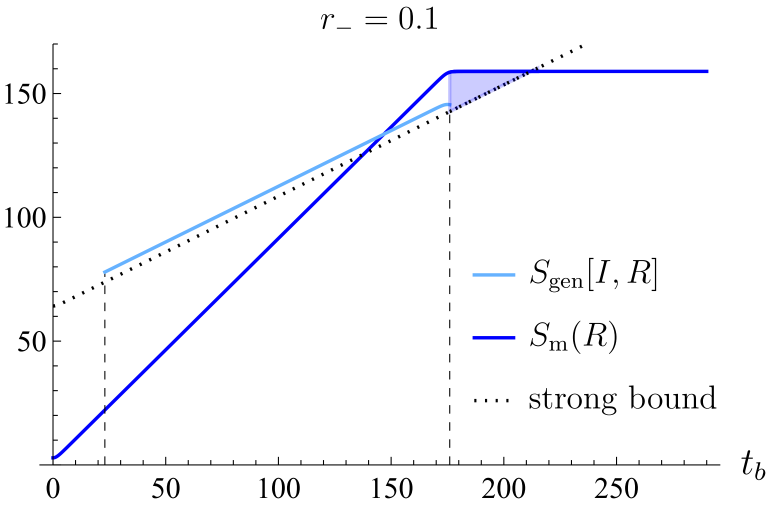
Numerical solution to the strong bound inequality with respect to the critical value of $r_-(q)$, at which the strong bound is satisfied in Reissner-Nordström black hole:

$$\frac{d}{dt_b}\left(S_{\mathrm{m}}(R) - S_{\mathrm{m}}(C)\right)\Big|_{t_b = t_{b,\max}} = 0, \quad (44)$$

$$S_{\mathrm{m}}(R) - S_{\mathrm{m}}(C)\Big|_{t_b = t_{b,\max}} \leq 2S_{\mathrm{BH}}. \quad (45)$$

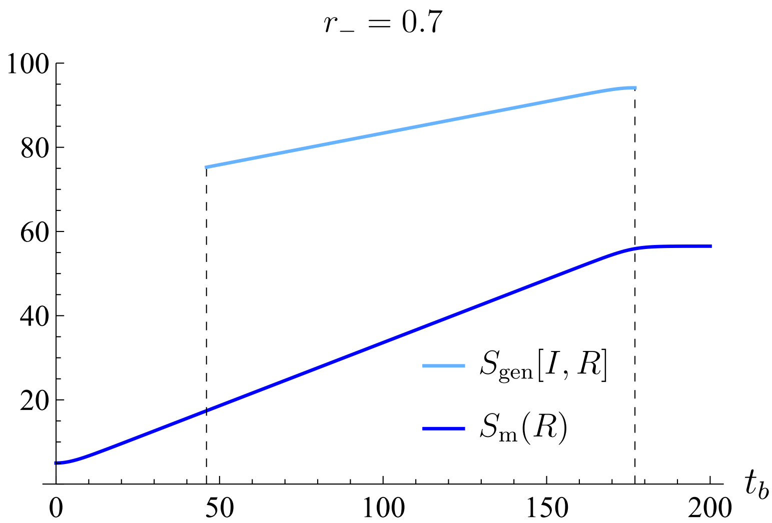
Lower values of q correspond to negative solutions $r_-(q) < 0$, so they are not considered.

Reissner-Nordström black hole: information paradox for finite regions



Evolution of $S_{\rm m}(R)$, $S^{\rm ext}_{\rm gen}[I, R]$ and the strong bound (dotted black) for a finite-size entangling region R, $r_- \ll r_+$. EE of Hawking radiation at each moment is given by the minimum of $S_{\rm m}(R)$ and $S^{\rm ext}_{\rm gen}[I, R]$. When the island disappears, there is a discontinuity. The larger the value of q is — the longer the time t_v of violation of the strong bound. Thus, the island prescription does not resolve the information paradox completely in the given setup.

Reissner-Nordström black hole: information paradox for finite regions



Evolution of $S_{\text{m}}(R)$, $S_{\text{gen}}^{\text{ext}}[I, R]$ and the strong bound (dotted black) for a finite-size entangling region R, $r_- \sim r_+$. EE of Hawking radiation at each moment is given by the minimum of $S_{\text{m}}(R)$ and $S_{\text{gen}}^{\text{ext}}[I, R]$. Starting from some r_-, the dominant contribution comes from the entropy of matter $S_{\text{m}}(R)$. Thus, the strong bound in Reissner-Nordström black hole is obeyed starting from $r_{-,\text{bound}}$ for a given q, regardless whether we consider islands or not.

Reissner-Nordström black hole: why no information paradox?

Why EE of Hawking radiation does not exceed the strong bound in near-extremal Reissner-Nordström black hole starting from some finite size of the entangling region?

- Hawking radiation is thermal $\implies$ Stefan-Boltzmann law:

$$S_{\text{thermo}}(R) \propto V(R) T_H^3, \tag{46}$$

  where $V(R)$ is the volume of the region R.
- $S_{\text{thermo}}(R)$ limits from above the fine-grained entropy $S_{\text{m}}(R)$ and, in turn, should not exceed S_{BH}:

$$S_{\text{m}}(R) \leq S_{\text{thermo}}(R) \leq S_{\text{BH}}. \tag{47}$$

Reissner-Nordström black hole: why no information paradox?

- In Schwarzschild black hole, this inequalities explain why for sufficiently small regions ($q \gtrsim b$) there is no information paradox: the thermodynamic entropy is proportional to the volume of the region $V(R)$, which can be made small enough so the strong bound is satisfied.

- In Reissner-Nordström black hole near extremality, $T_H \to 0$, and thermodynamic entropy $S_{\text{thermo}}(R)$ decreases, while S_{BH} does not change significantly ($r_+ \to r_h/2$). By choosing a sufficiently low temperature (or, equivalently, sufficiently large electric charge Q), we can ensure that the strong bound is obeyed without involving the island formula.

De Sitter space: difficulties

- In de Sitter space (dS), event horizon lies between the static observer and the null infinity.
- Therefore, there is no unambiguous way to couple the thermal bath in dS.
- Even if the bath is coupled to asymptotic infinity, the Euclidean gravitational path integral will not obtain the corresponding contribution, since only the static patch survives in Euclidean signature.
- Therefore, the problem of calculating EE in dS does not currently have a solution neither from holography nor from replica calculation.
- Attempts have been made to study EE in $d = 2$ JT gravity with a positive cosmological constant [*Hartman et al.'20; Balasubramanian et al.'20*]. However, the above reasons are an obstacle to a complete understanding even in $d = 2$.

De Sitter space: geometry

The metric of dS_3 is given by:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\phi^2, \quad f(r) = 1 - \frac{r^2}{\ell^2}. \tag{48}$$

The radial distance $d(\mathbf{x}, \mathbf{y})$ between two points reads:

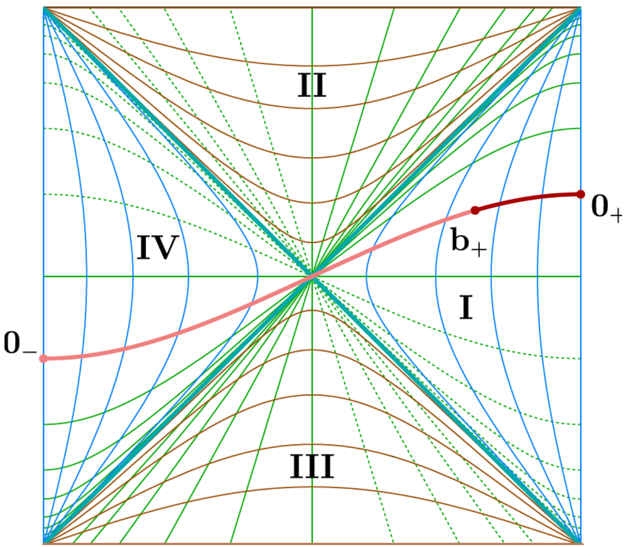
$$d^2(\mathbf{x}, \mathbf{y}) = \frac{2\sqrt{f(x)f(y)}}{\kappa_c^2}\Big[\cosh \kappa_c(r_*(x) - r_*(y)) - \cosh \kappa_c(t_x - t_y)\Big], \tag{49}$$

where $\kappa_c = 1/\ell$ is surface gravity, and the tortoise coordinate $r_*(r)$ is

$$r_*(r) = \frac{\ell}{2}\log\frac{\ell + r}{|\ell - r|} = \begin{cases} \ell \operatorname{arctanh} r/\ell, & r < \ell, \\ \ell \operatorname{arctanh} \ell/r, & r > \ell. \end{cases} \tag{50}$$

In calculations we use partial reduction from $d = 3$ to $d = 2$:
$\mathcal{S}_{\text{pure dS}_4} \to \mathcal{S}_{\text{partially reduced dS}_2} + \mathcal{S}_{\text{CFT}_2\text{ fermions}}$.

De Sitter space: no purity



- Cauchy surfaces Σ are finite-sized $\implies$ no need IR regularization.
- Basic properties of EE are violated in the background of partially reduced dS_3, namely:
  - $S_{\mathrm{m}}(\Sigma) \geq \dfrac{c}{3} \log \dfrac{2}{\kappa_c \varepsilon} \neq 0$,
  - $S_{\mathrm{m}}(R) \neq S_{\mathrm{m}}\left(\overline{R}\right)$.

De Sitter space: no complememtarity

Let us consider a finite region $R = [\mathbf{b}_+, \mathbf{0}_+]$.

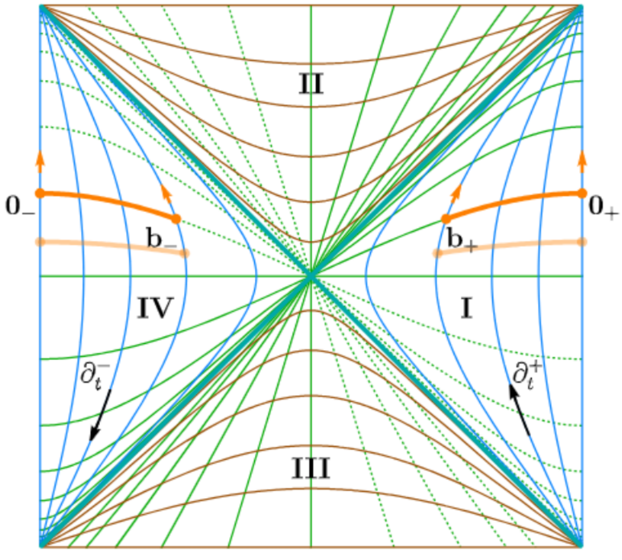
- EE for a finite interval and its complement are given by:

$$S_{\mathrm{m}}(R) = \frac{c}{6}\log\left(\frac{2\sqrt{f(b)}}{\kappa_c^2\varepsilon^2}\left[\cosh\kappa_c r_*(b) - \cosh\kappa_c(t_b - t_{0_+})\right]\right), \tag{51}$$

$$S_{\mathrm{m}}\left(\overline{R}\right) = \frac{c}{6}\log\left(\frac{2\sqrt{f(b)}}{\kappa_c^2\varepsilon^2}\left[\cosh\kappa_c r_*(b) + \cosh\kappa_c(t_b - t_{0_-})\right]\right). \tag{52}$$

- If $\kappa_c(t_b - t_{0_+}) =$ const, then $S_{\mathrm{m}}(R) =$ const, and $S_{\mathrm{m}}(R) \leq S_{\mathrm{GH}} \propto$ Area(horizon).
- Let $\kappa_c(t_b - t_{0_+}) \equiv g(t_b) \neq$ const. Then:
  - If $g'(t_b) > 0$, then $S_{\mathrm{m}}(R) \to \infty$, and the problem becomes ill-defined.
  - If $g'(t_b) < 0$, the endpoints $\mathbf{b}_+$ and $\mathbf{0}_+$ asymptotically approach the hypersurface of constant time at late times, and $S_{\mathrm{m}}(R) \to$ const.

De Sitter space: no islands



- At early times: $S_{\mathrm{m}}\left(R\right) \simeq \dfrac{2c}{3}\kappa_c t_b.$
- At late times: $S_{\mathrm{m}}\left(R\right) \simeq \text{const}.$
- As $b \to \ell$, the linear growth regime gets longer. This might formally lead to the information paradox in dS.
- Numerical analysis reveals no island for this configuration, so this possible violation of unitarity cannot be resolved with the island formula in the described setup.

Conclusions

- As a consistency test of the validity of studying EE in $d > 2$ by the CFT_2 formulas in $d = 2$ backgrounds obtained by either s-wave approximation or partial reduction of higher-dimensional geometries, we studied basic properties of EE for pure vacuum states in spherically-symmetric $d = 4$ black holes (s-wave approximation) and in dS_3 (partial reduction).
- In the case of black holes, purity and complementarity are violated by a constant. However, since this constant is the same, we prescribe to renormalize EE by subtracting it after a proper IR regularization of EE in a way that the basic properties of EE are satisfied in explicit calculations.
- In the case of partially reduced dS_3, both purity and complementarity are also violated. Since EE in this geometry is IR-finite, we cannot satisfy the basic properties of EE in explicit calculations by a proper IR regularization. This fact disproves CFT_2 approach in the described setup.

Conclusions

- The island formula does not fully resolve the information paradox for finite regions in black hole geometries. Namely, it predicts unnatural behavior of EE with discontinuity at the moment when entanglement island disappears as a solution of extremization equations in the descibed setup. This raises the question of either consistency of the s-wave approximation in studying EE in higher-dimensional spherically-symmetric black hole geometries or the validity of the island prescription. This is a subject of future research.
- In partially reduced dS_3, we give an example of finite entangling region, which might potentially lead to the information paradox. Numerical calculations reveal no non-trivial island for this configuration. The possible information paradox cannot be resolved by the island formula. Future investigations are needed to clarify the island prescription for dS.

Thank you for your attention!