# Slow-roll approximations in Einstein–Gauss–Bonnet gravity formulated in terms of e-folding numbers

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The $R + R^2$ gravity is the earliest inflationary model [1] which is in good agreement with modern observations[2]
The $R + R^2$ and the Higgs-driven inflation [3] belong cosmological attractors models [4] with inflationary parameters

$$n_s \simeq 1 - \frac{2}{N + N_0}, \quad r \simeq \frac{12 C_\alpha}{(N + N_0)^2}, \quad C_\alpha, \quad N_0 \ll 60 \quad \text{are constants} \tag{1}$$

the case $C_\alpha = 1$ corresponds to $R + R^2$ gravity [5].

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[1] A. A. Starobinsky, Phys. Lett. B **91** (1980), 99-102

[2] P. A. R. Ade *et al.* [BICEP and Keck], Phys. Rev. Lett. **127** (2021) no.15, 151301, arXiv:2110.00483 [astro-ph.CO]; Y. Akrami *et al.* [Planck], Astron. Astrophys. **641** (2020), A10, arXiv:1807.06211 [astro-ph.CO].

[3] F. L. Bezrukov and M. Shaposhnikov, Phys. Lett. B **659** (2008), 703-706, arXiv:0710.3755 [hep-th]

[4] M. Galante, R. Kallosh, A. Linde and D. Roest, Phys. Rev. Lett. **114** (2015) no.14, 141302, arXiv:1412.3797 [hep-th]

[5] The relation between time derivative and e-folding number derivative is $\dfrac{d}{dt} = -H\dfrac{d}{dN}$ ($H = -\frac{dN}{dt}$, $N = -\ln(a/a_0)$, $a_0$ is any number), the inflation interval is $0 < N < 65$ e-folding numbers

• Early we use the cosmological attractor inflationary parameters to reconstruct scenarios in Einstein-Gauss-Bonnet gravity[6] due to application of standard slow-roll regime.

• In the Einstein–Gauss–Bonnet (EGB) gravity models, the slow-roll approximation has been extended [7]by taking into account the first-order slow-roll parameter $\delta_1 = -2\,H^2\,\xi'/U_0$, which is proportional to the first derivative of the Gauss-Bonnet coupling function $\xi$ with respect to the e-folding number.

• These extensions lead to the question of the accuracy of effective potential reconstruction during the generalization of attractors in EGB gravity.

• We have reconstructed models using the extended slow-roll approximations and compared them with the exact expressions and the standard slow-roll approximation.

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[6]E. O. Pozdeeva, Eur. Phys. J. C **80** (2020) no.7, 612, arXiv:2005.10133 [gr-qc]. E. O. Pozdeeva and S. Y. Vernov, Eur. Phys. J. C **81** (2021) no.7, 633, arXiv:2104.04995 [gr-qc]

[7]E. O. Pozdeeva, M. A. Skugoreva, A. V. Toporensky and S. Y. Vernov, JCAP **09**, 050 (2024) doi:10.1088/1475-7516/2024/09/050 [arXiv:2403.06147 [gr-qc]].

# EGB gravity

The Einstein-Gauss-Bonnet gravity described by the following action:

$$S = \int d^4x\sqrt{-g}\left[U_0 R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) - \frac{1}{2}\xi(\phi)\mathcal{G}\right], \qquad (2)$$

where $U_0 > 0$ is a constant, the functions $V(\phi)$ and $\xi(\phi)$ are differentiable ones,
$R$ is the Ricci scalar,
$\mathcal{G} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ is the Gauss–Bonnet term.
Varying the action 2 with respect to the metric and considering the obtained equations in the spatially flat Friedmann space

$$ds^2 = -dt^2 + a^2(dx^2 + dy^2 + dz^2),$$

We formulate the Einstein equations in terms of e-folding number derivatives:

$$12\,Q\,\left(U_0 + 2\xi'\,Q\right) = Q\Phi + 2\,V, \tag{3}$$

$$2\,(Q)'\,(U_0 + 2\xi'\,Q) = Q\Phi - 4Q\left(Q\xi'' + \left(\frac{(Q)'}{2} + Q\right)\xi'\right), \tag{4}$$

where $' = \frac{d}{dN}$, $\Phi = \chi^2$, $\chi = \frac{d\phi}{dN}$, $Q = H^2$.
Varying of action (2) with respect to field and working in the spatially flat Friedmann Universe, reformulating obtained equation in terms of e-folding derivative, we get:

$$\frac{Q}{2}\Phi' + \frac{Q'}{2}\Phi - 3\,Q\,\Phi = -V' - 12\,Q\,\xi'\left(-\frac{Q'}{2} + Q\right). \tag{5}$$

The first slow-roll parameters

$$\epsilon_1 \quad = \quad \frac{1}{2}\frac{d\ln(Q)}{dN}, \;\; \delta_1 = -\frac{2\,Q}{U_0}\frac{d\xi}{dN}, \tag{6}$$

were early introduced in terms of time derivatives [8]
In the slow-roll regime ($\ddot{\phi} \ll 3H\dot{\phi}$ or equivalently
$\ddot{\phi}\phi' = \frac{Q}{2}\Phi' + \frac{Q'}{2}\Phi \ll 3\,Q\,\Phi$) the equation (5) can be reduced to

$$3\,Q\,\Phi = V' + 12\,Q^2\,\xi'\,(1-\epsilon_1) \tag{7}$$

leading to

$$\Phi_{1,2} = \frac{V'_{1,2} + 12\,Q^2\,\xi'\,(1-\epsilon)}{3Q} \tag{8}$$

for the extended slow-roll approximations and

$$\Phi_{sl} = \frac{V'_{sl} + 12\,Q^2\,\xi'}{3Q} \tag{9}$$

for the standard slow-roll approximation ($\frac{Q'}{2} \ll Q$ or $\epsilon_1 \ll 1$).

[8] Z. K. Guo and D. J. Schwarz, Phys. Rev. D **81** (2010), 123520, arXiv:1001.1897 [hep-th]
C. van de Bruck and C. Longden, Phys. Rev. D **93** (2016) no.6, 063519, arXiv:1512.04768 [hep-ph]

The first Friedmann equation can be presented in the form:

$$6\,U_0\,H^2\,(1-\delta_1) = \frac{\dot{\phi}^2}{2} + V. \qquad (10)$$

Supposition $\delta_1 \ll 1$ and $\dot{\phi}^2/2 \ll 6U_0H^2$ leads to the standard slow-roll approximation:

$$V \approx 6\,U_0\,Q. \qquad (11)$$

The slow-roll parameter $\delta_1$ grows during inflation and can becomes equal to one before end of inflation and can be taken into account considering slow-roll approximation of (3):

$$6U_0\,Q(1-\delta_1) \approx\; V. \qquad (12)$$

From here we get the potential approximations (16) and (15) using relation

$$(1-\delta_1) \approx \frac{1}{1+\delta_1}. \qquad (13)$$

Thus, the standard slow-roll approximation supposes:

$$V_{sl} = 6\, U_0\, Q. \tag{14}$$

The expressions of the potential for the expanded slow-roll approximations are :

$$\begin{aligned} V_1 &\approx \frac{6\, U_0\, Q}{1+\delta_1}, && (15) \\ V_2 &\approx 6\, U_0\, (1-\delta_1)\, Q. && (16) \end{aligned}$$

The obtained evolution equations allows to get expression for $\Phi = \chi^2$ using (4)

$$\Phi_{exact} = \frac{2\,U_0\,Q' + 6\,Q\,Q'\,\xi' + 4\,Q^2\,(\xi'' + \xi')}{Q} \tag{17}$$

and for the potential $V$ substituting (17) to (3)

$$V_{exact} = -\,(3\,\xi'\,Q + U_0)\;Q' + 2\;(5\,\xi' - \xi'')\,Q^2 + 6\,U_0\,Q. \tag{18}$$

Substituting (16) to $\Phi_2$ from (8) and using the definition of the slow-roll parameter $\delta_1$ in terms of e-folding number derivatives (6), we get (17), thus $\Phi_2 = \Phi_{exact}$.

The expressions for the inflationary parameters: the tensor-to-scalar ratio, the spectral index of scalar perturbation and the amplitude of scalar perturbation were early formulated in EGB gravity [9]:

$$
\begin{aligned}
r &\approx 8|2\epsilon_1 - \delta_1| \qquad (19)\\
n_s &\approx 1 - 2\epsilon_1 + \frac{d\ln(r)}{dN} \qquad (20)\\
A_s &\approx \frac{Q}{\pi^2\, U_0\, r}. \qquad (21)
\end{aligned}
$$

[9] J.-c. Hwang and H. Noh,Phys. Rev. D (2005) [arXiv:gr-qc/0412126 [gr-qc]]; J.-c. Hwang and H. Noh, Phys. Rev. D (2005) [arXiv:gr-qc/0412126 [gr-qc]]; Z. K. Guo and D. J. Schwarz, Phys. Rev. D (2010) [arXiv:1001.1897 [hep-th]], C. van de Bruck and C. Longden, Phys. Rev. D (2016) [arXiv:1512.04768 [hep-ph]]

We start our reconstruction from (19) using (1) to get equation

$$8|2\epsilon_1 - \delta_1| = \frac{12C_\alpha}{(N+N_0)^2} \tag{22}$$

with the following form of solution

$$Q = Q_0 \exp\left(-\frac{3}{2}\frac{C_\beta}{(N+N_0)}\right),\ \xi = \frac{\xi_0\, Q_0}{Q}. \tag{23}$$

where $C_\beta$, $N_0$, $Q_0$, $\xi_0$ are model constants.
We substitute these expressions for $Q$ and $\xi$ into the formulas for the first slow-roll parameters (6):

$$\epsilon_1 = \frac{3}{4}\frac{C_\beta}{(N+N_0)^2},\ \ \delta_1 \quad = \quad \frac{3\,\xi_0\, Q_0}{U_0}\frac{C_\beta}{(N+N_0)^2}. \tag{24}$$

To get exit from inflation at $N = 0$ ($\epsilon_1 = 1$) we put $C_\beta = \frac{4\,N_0^2}{3}$.
We obtain following expressions for the inflationary parameters: the tensor-to-scalar ratio, the spectral index of scalar perturbation and the amplitude of scalar perturbation:

$$
\begin{aligned}
r &\approx 8|2\epsilon_1 - \delta_1| \approx \frac{16\,N_0^2}{(N+N_0)^2}\left|1 - \frac{4\,\xi_0\,Q_0}{U_0}\right|, && (25)\\
n_s &\approx 1 - 2\epsilon_1 + \frac{d\ln(r)}{dN} \approx 1 - \frac{2}{N+N_0}\left(1 + \frac{N_0}{N+N_0}\right), && (26)\\
A_s &\approx \frac{Q}{\pi^2\,U_0\,r} \approx \frac{Q_0(N+N_0)^2\exp\left(-\frac{2\,N_0^2}{(N+N_0)}\right)}{16\pi^2\,U_0\,N_0^2\left|1 - \frac{4\,\xi_0\,Q_0}{U_0}\right|}. && (27)
\end{aligned}
$$

We derive the expressions for the potential of the exponential model under consideration using approximations and the exact formula:

$$V_{sl} = 6 U_0 Q_0 \exp\left(-\frac{2{N_0}^2}{N + N_0}\right), \tag{28}$$

$$V_1 = V_{sl}\left(1 + \frac{4 Q_0\, \xi_0\, {N_0}^2}{U_0\, (N + N_0)^2}\right)^{-1}, \tag{29}$$

$$V_2 = V_{sl}\left(1 - \frac{4\, Q_0\, \xi_0\, {N_0}^2}{U_0\, (N + N_0)^2}\right), \tag{30}$$

$$V_{exact} = V_{sl}\left(1 - \frac{N_0^2}{3\,(N + N_0)^2} - \frac{2 Q_0 \xi_0 {N_0}^2 (5\,(N + N_0)^2 - N_0^2 + 2\,(N + N_0))}{3\, U_0\, (N + N_0)^4}\right)$$

We substitute the potentials to corresponding expressions for $\Phi = (\phi')^2$ and get:

$$\Phi_{sl} = \frac{4\,{N_0}^2\,(U_0 - 2\,Q_0\,\xi_0)}{(N+N_0)^2}, \tag{31}$$

$$\Phi_1 = \frac{4\,U_0^2\,N_0^2\left(1 + \frac{4\xi_0\,Q_0(N+N_0)}{U_0(N+N_0)^2+4\xi_0\,Q_0\,N_0^2}\right)}{U_0(N+N_0)^2 + 4\xi_0\,Q_0\,N_0^2} + \frac{8\,Q_0\xi_0\,N_0^2\left(\frac{N_0^2}{(N+N_0)^2} - 1\right)}{(N+N_0)^2}, \tag{32}$$

$$\Phi_2 = \frac{4\,N_0^2}{(N+N_0)^2}\left(U_0 - 2\,Q_0\xi_0 + \frac{4\,Q_0\,\xi_0}{(N+N_0)} - \frac{2\,Q_0\xi_0\,N_0^2}{(N+N_0)^2}\right), \tag{33}$$

$$\Phi_{exact} = \Phi_2 \tag{34}$$

The field $\phi$ can be expressed analytically for the exponential model. To avoid complications during subsequent numerical calculations, it is better to integrate $\sqrt{\Phi_{exact}}$ after selecting the model parameters.
The expression $\Phi_1$ is very long and does not lead to an analytical expression for $\phi_1(N)$.
In the standard slow-roll approximation, the expression for the field's dependence on the e-folding number during inflation has a simple form

$$\phi_{sl} = 2\,N_0\,\sqrt{U_0 - 2\,Q_0\,\xi_0}\,\ln\left(N + N_0\right) + c_{sl}, \tag{35}$$

where $c_{sl}$ is constant, which should be chosen in order to compare the behavior of $\phi$ and $\phi_{sl}$ in numerical analysis.

# Choice of model parameters, numerical estimation

To analyze the behavior of $\phi(N)$ we should choose the model parameters, from eq. (25) we get:

$$\left|1 - \frac{4\xi_0 Q_0}{U_0}\right| = \frac{3}{4}\frac{C_\alpha}{N_0^2}, \qquad (36)$$

where $C_\alpha$ is a constant . At the choice (36), the expression for the scalar perturbation amplitude at the beginning of inflation $(N = N_b)$ is

$$A_s|_{N=N_b} = \frac{Q_0\,(N_b + N_0)^2\,\exp\left(-\frac{2\,N_0^2}{N_b+N_0}\right)}{12\pi^2\,U_0^2\,C_\alpha}, \qquad (37)$$

from where we can determine the value of $Q_0$:

$$Q_0 = \frac{12\pi^2\,U_0^2\,C_\alpha \cdot (A_s|_{N=N_b}) \cdot \exp\left(\frac{2\,N_0^2}{N_b+N_0}\right)}{(N_b + N_0)^2}. \qquad (38)$$

Fixing $N_0^2$, $C_\alpha$ and $Q_0$ we choose $\xi_0$, such as $\xi_0 = \frac{U_0}{4Q_0}\left(1 - \frac{3}{4}\frac{C_\alpha}{N_0^2}\right)$, where $\frac{4\xi_0 Q_0}{U_0} < 1$.

Now we start numerical estimations. We define the constant coupling $U_0$ as $U_0 = M_{Pl}^2/2$. For convenience, we put $N_0 = 1$. For simplicity, we follow to estimation for tensor-to-scalar ratio $r$ corresponding to $R + R^2$ inflationary model using $C_\alpha = 1$.

Using the observations of the cosmic microwaves background

$$A_s = (2.10 \pm 0.03) \times 10^{-9}, n_s = 0.9654 \pm 0.0040, r < 0.028$$

[10], we introduce values of the model parameters. We estimate the starting point of inflation, $N = N_b$, by substituting the spectral index value $n_s = 0.9654$ into equation (26):

$$N_b \approx 57.787. \tag{39}$$

We apply expression for $Q_0$ (38) by substituting $A_s = 2.1 \cdot 10^{-9}$, and obtain:

$$Q_0 \approx 1.8861 \cdot 10^{-12}\pi^2. \tag{40}$$

We calculate $\xi_0 = \xi_{01}$ assuming that the expression in the module brackets is positive:

$$\xi_0 \approx 1.6788 \cdot 10^{10}/\pi^2. \tag{41}$$

For obtained model parameters, we get $r \approx 0.0035$ at the beginning of inflation.

[10] Planck 2018; BICEP 2021; Galloni 2022

Early we introduce the effective potential for EGB gravity [11]. The effective potential $V_{eff}$ for it's equivalent presentation can play the role of potential for studying stability [12]. Here, we consider an equivalent presentation of the effective potential $\tilde{V}_{eff}$:

$$\tilde{V}_{eff} = -V_{eff}^{-1}, \quad \text{where} \quad V_{eff} = -U_0^2/V + \xi/3. \tag{42}$$

Here we should note, that the considering set of the model parameters is mostly toy and used to clear present the behavior of effective potential $\tilde{V}_{eff}$ which is analog to potential in the General Relativity models.

[11] E. O. Pozdeeva, M. Sami, A. V. Toporensky and S. Y. Vernov, Phys. Rev. D (2019) [arXiv:1905.05085 [gr-qc]].

[12] S. Vernov and E. Pozdeeva, Universe (2021) [arXiv:2104.11111 [gr-qc]].

All approximations and exact behavior of $\tilde{V}_{eff}$ have good agreements during inflation and start deviate near $N = 5$. This deviation is mostly interesting after inflation, because after inflation only equivalent exact effective potential has potential well. The all approximations and exact expression of $\tilde{V}_{eff}$ are closed to zero after $N \approx -0.8$. The corresponding behaviors of the equivalent effective potential are presented in Fig.1. The exact $\tilde{V}_{eff}$ has minimum at $N_m \approx -0.4463$ and equals to $\tilde{V}_{eff}(N_m) \approx -4.4571 \cdot 10^{-12}$ (see the Fig.1). The equivalent effective potential $\tilde{V}_{eff} \approx 0$ if $N \approx -0.2392$ and $N \approx -0.7777$.

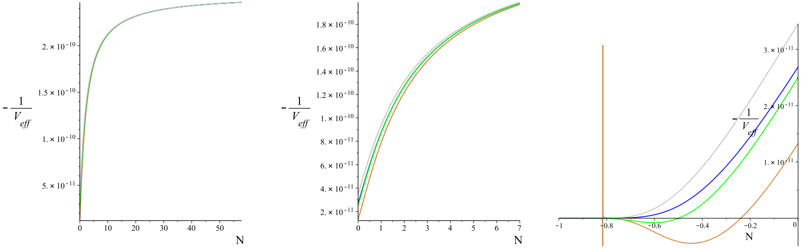


Figure: The behavior of $\tilde{V}_{eff}$ during inflation for slow-roll approximations (the gray line corresponds to the standard slow-roll approximation, the blue line to the approximation (15), the green line to the approximation (16)) and the exact considerations (orange line) at the following values of the parameters: $N_0 = 1$, $N_b = 57.787$, $\xi_0 = 1.6788 \cdot 10^{10}/\pi^2$, $Q_0 \approx 1.8861 \cdot 10^{-12}\pi^2$, $U_0 = M_{Pl}^2/2$, $M_{Pl} = 1$.

Using the values of the model parameters, we obtain:

$$\Phi = (\phi')^2 \approx \frac{1.75\,(N+1.5469)\,(N+0.73880)}{(N+1)^4}, \qquad (43)$$

$$\phi = 0.5\arctan\left(\frac{0.37796(N-1.7202\cdot 10^{-9})}{S_q}\right) \qquad (44)$$

$$+ \quad 1.3229\ln((N+1.1429+S_q)\cdot 10^9) - \frac{1.3229\,S_q}{(N+1)} - c_0, \qquad (45)$$

where $S_q = \sqrt{(N+1.5468)\cdot(N+0.73886)}$, $c_0$ is a constant of integration. The choice of integration constant $c_0$ is not unique. Near $N \approx -0.738$ the field becomes complex and we choose $c_0 \approx 25.431$ to get $\phi|_{N\approx -0.738} = 0$.

We calculate the integration constant included to the slow-roll approximation of the field $c_{sl} = \phi - (2\,N_0\sqrt{U_0 - 2\,Q_0\,\xi_0}\ln(N+N_0))$ at the point $N = N_b$ and get: $c_{sl} \approx 1.75$. The choice of the integration constant allows us fix the same values of fields at the beginning of inflation. We solve the differential equation $\frac{d\phi}{dN} = \sqrt{\Phi}$ numerically assuming $\phi(N = N_b) = 7.145$ for all types of approximations and the exact solution.

In Fig. 2, the dependence of the equivalent effective potential $\tilde{V}_{eff}$ on the field for all type of approximation and exact case are presented.

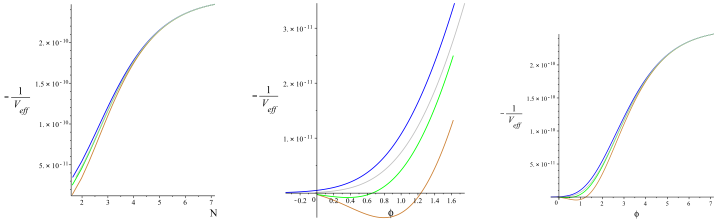


Figure: The graphical behavior of effective potential $\tilde{V}_{eff} = -V_{eff}^{-1}$ during inflation for slow-roll approximations (the gray line corresponds to the standard slow-roll approximation, the blue line to the approximation (15), the green line to the approximation (16) ) and the exact considerations (orange line) at the following values of the parameters: $c_{sl} = 1.7556$, $N_0 = 1$, $N_b = 57.787$, $\xi_0 = 1.6569 \cdot 10^{10}/\pi^2$, $Q_0 \approx 1.8861 \cdot 10^{-12}\pi^2$, $U_0 = M_{Pl}^2/2$, $M_{Pl} = 1$. The left picture corresponds to evolution during inflation, the middle picture - to evolution after inflation, the right picture is the join evolution.

# Conclusion

•The standard and the extended slow-roll approximations were considered and verified using the exact solution for the exponential model of EGB gravity.

• The extended slow-roll approximation with $V \sim (1 - \delta_1)$ allows us to reconstruct the dependence of exact field on the e-folding numbers. However, the extended slow-roll approximation with $V \sim (1 + \delta_1)^{-1}$ does not lead to an analytical dependence of the field on the e-folding numbers.

• All potentials reconstructed in this study preserve the exponential nature of the potential obtained within the standard slow-roll approximation framework. The first terms of all potentials coincide with $V_{sl}$.

• The standard slow-roll approximation reproduces $\phi(N)$ of the exact model for $N$ between $N \approx 58$ and $N \approx 5$ up to an integration constant. After that, the deviation slowly increases before the end of inflation.

• The substitution of the standard slow-roll approximated $\phi(N)$ instead of the exact consideration to the models depending from field can change the total number of inflationary e-folding number in numerical tests. Despite this the field $\phi(N)$ approximated by standard slow-roll can be used to generate new EGB inflation models due to its simplicity in reasonable values of e-folding number.

• Both the standard and the extended slow-roll approximations reproduce exact effective potential up to $N \approx 5$ e-folding number with high accuracy.

• The effective potential obtained from the standard slow-roll approximation has bigger deviation from exact than obtained from the extended slow-roll approximations. We suppose that in models with difficult exact analytical considerations, the extended approximation with $V \sim (1 - \delta_1)$ can be applied with higher accuracy to describe inflation.

• The obtained results can be applied for generation of EGB gravity models. We hope that the results can be useful for better understanding of post-inflation evolution and for connection of early and later time processes.