Актуальные модели космологической инфляции

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Секция: Гравитация и космология

17-21 февраля 2025

Inflationary cosmological models

- Inflation $\Rightarrow$ cosmological perturbations $\Rightarrow$ large-scale structure formation (by scalar perturbations) and relic gravitational waves (tensor perturbations)
- Scalar and tensor perturbations $\Rightarrow$ CMB anisotropy and polarization $\Rightarrow$ observational constraints
- For the case of GR these constraints restrict the possible types of potentials of inflaton $V(\phi)$
- The additional constraints on the inflationary parameters besides the slow-roll approach
- For the case of modified gravity theories the models with arbitrary potentials can satisfy the observational constraints
- Restriction on modified gravity theories $|c_g - 1| \leq 5 \times 10^{-16}$ (gravitational waves from neutron star merging)

Friedmann-Robertson-Walker space-time

The standard cosmological theory of the big bang is based on the geometry of the Universe, which is homogeneous and isotropic at large distances, which is determined by the Friedmann-Robertson-Walker metric in the system of units $8\pi G = m_{pl}^{-2} = c = 1$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) \left(dx^2 + dy^2 + dz^2\right), \quad \mu, \nu = 0, 1, 2, 3, \tag{1}$$

where $a = a(t)$ is the scale factor, t is the cosmic time.
The case of $k = 0$ and $a(t) = \exp(\lambda t)$, where λ is some constant (exponential expansion) corresponds to the de Sitter metric

$$(ds^2)_{dS} = -dt^2 + \exp(2\lambda t) \left(dx^2 + dy^2 + dz^2\right); \tag{2}$$

Thus, a homogeneous isotropic four-dimensional Friedmann-Roberstson-Walker space is considered as a space-time model, which corresponds to the most common approach to describing the geometry of the Universe.

The action for the cosmological models

The action for the cosmological models

$$S = \int d^4x(\mathcal{L}_g + \mathcal{L}_m + \mathcal{L}_{coupl}). \qquad (3)$$

The gravitational part

$$\mathcal{L}_g = \Lambda + \frac{1}{2}R + \alpha_1 R^2 + \alpha_2 R_{\mu\nu}R^{\mu\nu} + \alpha_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \ldots \qquad (4)$$

Einstein gravity

$$\mathcal{L}_g = \frac{1}{2}R. \qquad (5)$$

Starobinsky gravity

$$\mathcal{L}_g = \frac{1}{2}R + \alpha_1 R^2. \qquad (6)$$

Einstein-Gauss-Bonnet gravity

$$\mathcal{L}_g = \frac{1}{2}R + R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{1}{2}R + G, \quad (\mathcal{L}_{coupl} \neq 0). \qquad (7)$$

Generalizations: $f = f(R)$, $f = f(G)$, including non-minimal coupling.
The other types of gravity theories...

Cosmological models based on GR

The action for inflationary models based on Einstein gravity and a single scalar field is (for $8\pi G = m_p^2 = c = 1$)

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right], \tag{8}$$

where ϕ is a scalar field, $V(\phi)$ is the potential of a scalar field and $g^{\mu\nu}$ is a metric tensor of a space-time. The corresponding dynamic equations in a spatially flat FWR space-time are

$$3H^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi) \equiv \rho_\phi, \tag{9}$$

$$-3H^2 - 2\dot{H} = \frac{1}{2}\dot{\phi}^2 - V(\phi) \equiv p_\phi, \tag{10}$$

$$\ddot{\phi} + 3H\dot{\phi} + V'_\phi = 0, \tag{11}$$

where ρ_ϕ and p_ϕ are the energy density and the pressure of a scalar field, $V'_\phi = dV/d\phi$.

The slow-roll approximation

The slow-roll conditions

$$H \simeq \lambda = const, \quad a(t) \approx \exp(\lambda t), \tag{12}$$

$$\frac{1}{2}\dot{\phi}^2 \ll V(\phi), \tag{13}$$

$$\epsilon \ll 1, \quad \delta \ll 1, \tag{14}$$

$$p_\phi \approx -\rho_\phi. \tag{15}$$

Approximate dynamic equations

$$3H^2 \approx V(\phi), \tag{16}$$

$$-2\dot{H} \approx \dot{\phi}. \tag{17}$$

Slow-roll parameters

$$\epsilon \simeq \frac{1}{2}\left(\frac{V'_\phi}{V}\right)^2, \tag{18}$$

$$\delta \simeq \frac{V''_\phi}{V} - \frac{1}{2}\left(\frac{V'_\phi}{V}\right)^2. \tag{19}$$

Cosmological parameters

When the slow-roll conditions $\epsilon \ll 1$ and $\delta \ll 1$ are fulfilled, the parameters of cosmological perturbations are

$$A_S = \frac{1}{2\epsilon_*}\left(\frac{H_*}{2\pi}\right)^2, \quad A_T = 8\left(\frac{H}{2\pi}\right)^2, \quad r = \frac{A_T}{A_S} = 16\epsilon_*, \quad n_S - 1 = -4\epsilon_* + 2\delta_*. \qquad (20)$$

Observational constraints on the parameters of cosmological perturbations

$$A_S = 2.1 \times 10^{-9}, \qquad n_s = 0.9649 \pm 0.0042, \qquad (21)$$

$$r < 0.028 \quad \text{(Planck 2018/BICEP2/Keck-Array)}. \qquad (22)$$

The difference in the e-folds number between the end of inflation and at the crossing of the Hubble

$$\Delta N \simeq 50 - 60. \qquad (23)$$

The Lyth bound

$$|\Delta\phi| \geq \mathcal{O}(1)\left(\frac{r}{0.01}\right)^{1/2}. \qquad (24)$$

The Swampland conjectures

$$|\Delta\phi| \leq \frac{1}{2}\ln\left(\frac{2}{\pi^2 A_S r}\right), \qquad \left|\frac{V'_\phi}{V}\right|_{\phi=\phi_*} \geq \mathcal{O}(1). \qquad (25)$$

Classification of inflationary models

The dependence $r = r(1 - n_S)$ can characterizes the type of inflationary model.
Since the value of the spectral index of scalar perturbations is $n_S \simeq 0.97$ and $1 - n_S \simeq 0.03 \ll 1$, we can write the dependence $r = r(1 - n_S)$ as follows

$$r = \sum_{k=0}^{\infty} \beta_k (1 - n_S)^k = \beta_0 + \beta_1 (1 - n_S) + \beta_2 (1 - n_S)^2 + ..., \qquad (26)$$

where $(1 - n_S)$ is the small parameter of expansion and β_k are the constant coefficients.
Since, the zeroth order term in this expansion $r(0) = \beta_0 = 0$ from condition $r(n_S = 1) = 0$ corresponding to the flat Harrison-Zel'dovich spectrum, one can rewrite expression (26) in the following form

$$r = \sum_{k=1}^{\infty} \beta_k (1 - n_S)^k = \beta_1 (1 - n_S) + \beta_2 (1 - n_S)^2 + ... = r^{(1)} + r^{(2)} + ... \qquad (27)$$

The expected difference between the values of the tensor-to-scalar ratio for the second-order and first-order inflationary models is

$$\frac{r^{(2)}}{r^{(1)}} \sim \frac{(1 - n_S)^2}{(1 - n_S)} \sim 10^{-2}. \qquad (28)$$

First-order models $\epsilon/\delta = const$

Chaotic inflation with quadratic potential

$$V(\phi) = \frac{m^2}{2}\phi^2, \tag{29}$$

where m is the mass of a scalar field.

$$\delta \simeq \frac{V''_\phi}{V} - \frac{1}{2}\left(\frac{V'_\phi}{V}\right)^2 = 0, \tag{30}$$

$$r = 4(1 - n_S). \tag{31}$$

Due to the observational value of the spectral tilt of the scalar perturbations $n_S \simeq 0.97$ one has $r \simeq 0.1$.

Minimal value of tensor-to-scalar ratio for the first-order inflationary models based on GR is $r^{(1)} \simeq 0.1$.

Expected value of the tensor-scalar ratio for second-order inflationary models based on GR is $r^{(2)} \simeq r^{(1)} \times 10^{-2} \sim 10^{-3}$.

First-order models $\delta/\epsilon = const$

The inflationary model with the Higgs potential

$$V(\phi) = \frac{\lambda_H}{4}\left(\phi^2 - \sigma^2\right)^2, \quad (32)$$

where λ_H is self-coupling constant, and σ is the vacuum expectation value of the Higgs field.

$$\delta \simeq \frac{1}{2}\epsilon. \quad (33)$$

$$r = \frac{16}{3}(1 - n_S). \quad (34)$$

For $n_S \simeq 0.97$ one has $r \simeq 0.16$.

$F(R)$ – gravity

$$S_F = \frac{1}{2}\int d^4x\sqrt{-g}\,F(R), \tag{35}$$

$$S_\chi = \frac{1}{2}\int d^4x\sqrt{-g}\;\left[F'(\chi)(R-\chi)+F(\chi)\right], \tag{36}$$

$$V = \frac{\chi F'(\chi)-F(\chi)}{2F'(\chi)^2}, \tag{37}$$

$$\frac{dV}{d\chi} = \frac{F''(\chi)\left[2F(\chi)-\chi F'(\chi)\right]}{2(F'(\chi))^3}, \tag{38}$$

$$F'(\chi) = \exp\left(\sqrt{\frac{2}{3}}\phi\right), \qquad \phi = \sqrt{\frac{3}{2}}\ln F'(\chi), \tag{39}$$

$$S_E = \int d^4x\sqrt{-g_E}\left[\frac{1}{2}R_E - \frac{1}{2}g_E^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]. \tag{40}$$

Starobinsky model

Relationships between the type of $f(R)$-gravity and the scalar field potential

$$R = \left[\sqrt{6}\,\frac{dV}{d\phi} + 4V\right]\exp\left(\sqrt{\frac{2}{3}}\phi\right), \tag{41}$$

$$F = \left[\sqrt{6}\,\frac{dV}{d\phi} + 2V\right]\exp\left(2\sqrt{\frac{2}{3}}\phi\right). \tag{42}$$

For Starobinsky gravity (in Jordan frame)

$$F(R) = R + \frac{1}{6m^2}R^2, \tag{43}$$

corresponding potential (in Einstein frame) is

$$V(\phi) = \frac{3}{4}m^2\left(1 - e^{-\sqrt{\frac{2}{3}}\phi}\right)^2. \tag{44}$$

A. A. Starobinsky, "A New Type of Isotropic Cosmological Models Without Singularity," Phys. Lett. B **91** (1980), 99-102 doi:10.1016/0370-2693(80)90670-X

S. S. Mishra, V. Sahni and A. V. Toporensky, "Initial conditions for Inflation in an FRW Universe," Phys. Rev. D **98** (2018) no.8, 083538 doi:10.1103/PhysRevD.98.083538

Starobinsky model as the cosmological model of the second order

Starobinsky inflation

$$V(\phi) = \frac{3}{4}m^2\left(1 - e^{-\sqrt{\frac{2}{3}}\phi}\right)^2. \tag{45}$$

$$r = 3(1 - n_S)^2 \simeq 0.003 \sim r^{(2)}. \tag{46}$$

α–attractors (as example α-Starobinsky potential)

$$V(\phi) = \frac{3}{4}m^2\left(1 - e^{-\sqrt{\frac{2}{3\alpha}}\phi}\right)^2. \tag{47}$$

In general case different potentials with following relation

$$r = 3\alpha(1 - n_S)^2, \tag{48}$$

$$3\alpha = 1, 2, 3, ..., 7, \tag{49}$$

$$r \sim 10^{-2} - 10^{-3}. \tag{50}$$

R. Kallosh, A. Linde and D. Roest, "Superconformal Inflationary α-Attractors," JHEP **11** (2013), 198 doi:10.1007/JHEP11(2013)198

Verified inflationary models based on Einstein gravity

The first order inflationary models based on Einstein gravity $r \sim (1 - n_S)$ don't verified by observational constraints.

Consequence $\Rightarrow$ The additional conditions on the inflationary parameters.

The slow-roll conditions:

$$0 < \epsilon \ll 1, \qquad |\delta| \ll 1. \tag{51}$$

The second order $r \sim (1 - n_S)^2$ conditions:

$$\left(\delta^2/\epsilon\right) \approx const, \quad \epsilon_* \ll |\delta_*|\,. \tag{52}$$

These conditions restrict the possible types of the scalar field potential $V = V(\phi)$.

Verified inflationary models for arbitrary scalar field potentials $\Rightarrow$ modified gravity theories.

Cosmological models based on the scalar-tensor gravity

The models of cosmological inflation with Einstein gravity

$$S_E = \int_{\mathcal{M}} d^4x\sqrt{-g}\left[\frac{1}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi_E\partial_\nu\phi_E - V_E(\phi_E)\right], \qquad (53)$$

The action for inflationary models based on scalar-tensor gravity theories

$$S_{STG} = \int d^4x\sqrt{-g}\left[\frac{1}{2}F(\phi)R - \frac{\omega(\phi)}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right], \qquad (54)$$

Connection between potential and STG parameters

$$H = \lambda\sqrt{F}, \qquad (55)$$

$$F(\phi) \simeq \frac{1}{\lambda}\sqrt{\frac{V(\phi)}{3}}, \qquad (56)$$

$$\omega(\phi) \simeq -\frac{\sqrt{3}(1+2k)}{24\lambda}\left(\frac{V_{\phi}^{\prime 2}}{V^{3/2}}\right), \qquad (57)$$

$$\lambda\sqrt{\frac{V(\phi)}{3}} \simeq \left[(2k-1)(\alpha t-\lambda)\right]^{2/(1-2k)}. \qquad (58)$$

I. V. Fomin, S. V. Chervon, A. N. Morozov and I. S. Golyak, Relic gravitational waves in verified inflationary models based on the generalized scalar–tensor gravity, Eur. Phys. J. C **82** (2022) no.7, 642
doi:10.1140/epjc/s10052-022-10601-9

Parameters of cosmological perturbations

$$\mathcal{P}_S = A_S = \frac{\lambda^2}{16\pi^2\epsilon(\epsilon-\delta)}, \tag{59}$$

$$\mathcal{P}_T = A_T = \frac{2\lambda^2}{\pi^2}, \tag{60}$$

$$r = \frac{\mathcal{P}_T}{\mathcal{P}_S} = 32\epsilon(\epsilon-\delta), \tag{61}$$

$$n_S = 1 - 4\epsilon + 2\delta + 2\epsilon\delta + (1-\epsilon)\left(\frac{\epsilon\delta-\xi}{\epsilon-\delta}\right), \tag{62}$$

$$n_T = 0. \tag{63}$$

$$r = \frac{2}{1-k}(1-n_S)^2, \qquad \text{Second-order models.} \tag{64}$$

From observational constraints one has

$$\lambda^2 < 6.7\times 10^{-10}, \tag{65}$$

$$k < 0.96. \tag{66}$$

The dependence $r = r(n_S)$

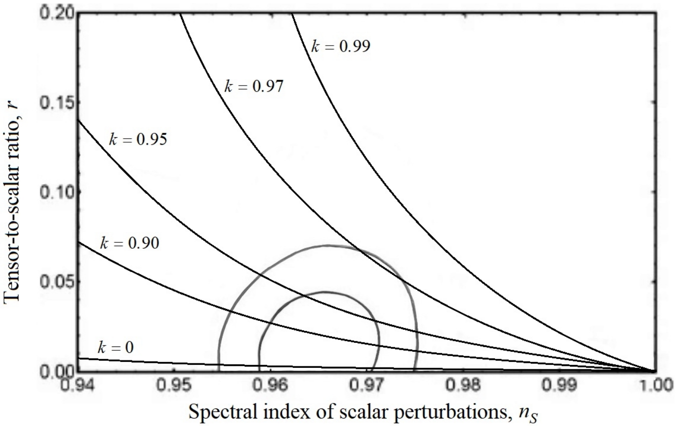

Рис.: The $r = r(n_S)$ dependence and observational constraints.

Chaotic inflation with quadratic potential

Chaotic inflation with quadratic potential

$$V(\phi) = \frac{1}{2}m^2\phi^2. \tag{67}$$

The coupling and kinetic functions are

$$F(\phi) = \phi, \tag{68}$$

$$\omega(\phi) = -\frac{1+2k}{2}, \tag{69}$$

corresponding to the Brans-Dicke gravity with $k < 1$, where $\lambda = \frac{m}{\sqrt{6}}$.

The cosmological models based on the EGB gravity

The models of cosmological inflation with Einstein gravity

$$S_E = \int d^4x\sqrt{-g}\left[\frac{1}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi_E\partial_\nu\phi_E - V_E(\phi_E)\right]. \qquad (70)$$

Inflationary models with additional non-minimal coupling of a scalar field and the Gauss-Bonnet term, the action is

$$S_{GB} = \int d^4x\sqrt{-g}\left[\frac{1}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi_{GB}\partial_\nu\phi_{GB} - V_{GB}(\phi_{GB}) - \frac{1}{2}\xi(\phi_{GB})R^2_{\rm GB}\right]. \qquad (71)$$

Index "E" denotes Einstein gravity, and index "GB" means Einstein-Gauss-Bonnet gravity.

The relation between the Hubble parameters for standard H_E and EGB inflation H_{GB} is

$$H_E = H_{GB}\left(1 - 2\dot{\xi}H_{GB}\right). \qquad (72)$$

I. Fomin, "Gauss–Bonnet term corrections in scalar field cosmology," Eur. Phys. J. C **80** (2020) no.12, 1145
doi:10.1140/epjc/s10052-020-08718-w

The conditions of a weak GB coupling

- A weak influence of non-minimal coupling of a scalar field and the Gauss-Bonnet scalar on cosmological dynamics

$$H_{GB} = H_E + \Delta_H, \quad \Delta_H \ll H_E. \tag{73}$$

- A weak GB coupling does not change the type (shape) of a scalar field potential V_E (such a coupling does not change the specificity of inflationary processes in comparison with standard inflationary models)

$$\Delta_H \equiv -\alpha_{GB}\frac{\dot{H}_E}{H_E}, \tag{74}$$

where $\alpha_{GB} < 1$ is a coupling constant.

- Consequence $\Rightarrow$ For a pure exponential (de Sitter) expansion of the universe $H_E = const$ one has no difference between standard inflation and EGB inflation with this type of a weak GB coupling: $\Delta_H = 0$ and $H_{GB} = H_E = const$.

The inflationary models with a weak GB coupling

$$\Delta_H \equiv -\alpha_{GB}\frac{\dot{H}_E}{H_E}. \tag{75}$$

For the quasi de Sitter stage

$$V_{GB}(\phi_{GB}) \simeq V_E(\phi_E), \tag{76}$$

$$H_{GB} \simeq H_E, \tag{77}$$

$$\phi_{GB} = \phi_E\sqrt{1-\alpha_{GB}}, \tag{78}$$

$$\xi(\phi_{GB}) = \frac{\alpha_{GB}}{4H_E^2(\phi_{GB})}. \tag{79}$$

The parameters of cosmological perturbations are

$$n_{S(GB)} - 1 = -4\epsilon_E + 2\delta_E = n_{S(E)} - 1, \tag{80}$$

$$r_{GB} = 16(1-\alpha_{GB})\epsilon_E = (1-\alpha_{GB})r_E. \tag{81}$$

The influence of the Gauss-Bonnet scalar on the inflationary parameters for such type of a weak GB coupling is determined by the value of the coupling constant α_{GB}.
Consequence $\Rightarrow$ The value of the coupling constant α_{GB} can be estimated by the observational constraints on the values of cosmological perturbations parameters.

Chaotic inflation with quadratic potential

Chaotic inflation with quadratic potential

$$V_{GB}(\phi_{GB}) = \frac{m_{GB}^2}{2}\phi_{GB}^2, \tag{82}$$

where m_{GB} is the mass of a scalar field.

$$r = 4(1-\alpha_{GB})(1-n_S), \qquad \text{First-order model} \tag{83}$$

Chaotic inflation with quadratic potential corresponds to the observational constraints for $0.45 < \alpha_{GB} < 1$.

The mass of the scalar field non-minimally coupled with the Gauss-Bonnet term is $m_{GB} > 1.35\, m_E$.
Field itself changes as $0 < |\Delta\phi_{GB}| < 0.74\, |\Delta\phi_E|$.

Conclusion

- Classification of inflationary models by the orders of expansion of dependence $r = r(1 - n_S)$
- Inflationary models based on GR are verifiable at the second order $r \sim (1 - n_S)^2$ for specific scalar field potentials
- Inflationary models based on modified gravity theories can be verifiable at the first order $r \sim (1 - n_S)$ and the second order $r \sim (1 - n_S)^2$ for arbitrary potentials of a scalar field by specific relations between modified gravity parameters

THANK YOU FOR ATTENTION!