Slow-roll approximations for inflationary models with nonminimally coupled scalar fields

С.Ю. Вернов

НИИЯФ МГУ

доклад основан на статье
E.O. Pozdeeva, M.A. Skugoreva, A.V. Toporensky, S.Yu. Vernov,
arXiv:2502.13008

Сессия-конференция
"Физика фундаментальных взаимодействий",
посвящённая 70-летию со дня рождения В.А. Рубакова
Москва, Россия, 19.02.2025

MODELS WITH THE NONMINIMAL COUPLED SCALAR FIELDS

The action of a generic model with a nonminimally coupled scalar field ϕ,

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left(F(\phi)R - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2V(\phi)\right), \qquad (1)$$

includes the coupling function $F(\phi) > 0$ and the potential $V(\phi)$.
In the spatially flat Friedmann-Lemaître-Robertson-Walker metric with

$$ds^2 = -\,dt^2 + a^2(t)\,d\mathbf{x}^2\,,$$

the field equations are ($H = \frac{\dot a}{a}$)

$$3H^2F = \frac{1}{2}\dot\phi^2 + V - 3F_{,\phi}\dot\phi H, \qquad (2)$$

$$2\dot HF = -\,\dot\phi^2 + F_{,\phi}\dot\phi H - F_{,\phi\phi}\dot\phi^2 - F_{,\phi}\ddot\phi, \qquad (3)$$

$$\ddot\phi + 3H\dot\phi + V_{,\phi} - 3F_{,\phi}\left(\dot H + 2H^2\right) = 0, \qquad (4)$$

where dots denote the time derivatives and $A_{,\phi} = \frac{dA(\phi)}{d\phi}$ for any A.

It is suitable to present Eqs. (3) and (4) as the following dynamical system:

$$\begin{aligned}
\dot{\phi} =& \psi,\\
\dot{\psi} =& \frac{1}{E}\left\{ -3F_{,\phi}\left[F_{,\phi\phi}+1\right]\psi^2 + 3\left[F_{,\phi}^2 - 2F\right]H\psi - 2F\left[V_{,\phi} - 6F_{\phi}H^2\right]\right\},\\
\dot{H} =& \frac{1}{E}\left\{ -\left[F_{,\phi\phi}+1\right]\psi^2 + 4F_{,\phi}H\psi + F_{,\phi}\left[V_{,\phi} - 6F_{\phi}H^2\right]\right\},
\end{aligned}$$

where

$$E = 3F_{,\phi}^2 + 2F.$$

We consider the e-folding number $N = \ln(a/a_e)$, where a_e is a constant, as a measure of time during inflation. Using the relation $\frac{d}{dt} = H\frac{d}{dN}$ and introducing the function

$$\chi(N) = \frac{d\phi}{dN} = \frac{\dot{\phi}}{H},$$

we get Eq. (2) in the following form

$$H^2 = \frac{2V}{6F + 6F_{,\phi}\chi - \chi^2}. \tag{5}$$

Using Eq. (5), we eliminate H^2 and get the following second order system:

$$\begin{aligned} \frac{d\phi}{dN} =& \chi, \\ \frac{d\chi}{dN} =& \frac{1}{2EV}\left\{\left[2V\left(F_{,\phi\phi}+1\right)+F_{,\phi}V_{,\phi}\right]\chi^3\right. \\ &+ 2\left[FV_{,\phi} - F_{,\phi}V\left(3F_{,\phi\phi}+7\right) - 3F_{,\phi}^2V_{,\phi}\right]\chi^2 \\ &+ \left. 6\left[3F_{,\phi}^2V - 3FF_{,\phi}V_{,\phi} - 2FV\right]\chi + 12F\left(2F_{,\phi}V - FV_{,\phi}\right)\right\}. \end{aligned} \tag{6}$$

The potential V appears as the first derivative of its logarithm only.

Another way to get a dynamical system from Eqs. (2)–(4) is to rewritten them in the following form:

$$3M_{\rm Pl}^2 Y^2 = \frac{A}{2}\dot\phi^2 + V_{\rm eff}, \tag{7}$$

$$\dot Y = -\frac{A\sqrt{F}}{2M_{\rm Pl}^3}\,\dot\phi^2, \tag{8}$$

$$\ddot\phi = -3\sqrt{\frac{F}{M_{\rm Pl}^2}}\,Y\dot\phi - \frac{A_{,\phi}}{2A}\dot\phi^2 - \frac{V_{{\rm eff},\phi}}{A}, \tag{9}$$

where

$$V_{\rm eff}(\phi) = \frac{M_{\rm Pl}^4 V}{F^2}, \quad A(\phi) = \frac{M_{\rm Pl}^4}{F^2}\left(1 + \frac{3F_{,\phi}^2}{2F}\right), \tag{10}$$

$$Y = \frac{M_{\rm Pl}}{\sqrt{F}}\left(H + \frac{F_{,\phi}\dot\phi}{2F}\right). \tag{11}$$

M.A. Skugoreva, A.V. Toporensky and S.Yu. Vernov, Phys. Rev. D **90** (2014) 064044 [arXiv:1404.6226],
A.Yu. Kamenshchik, E.O. Pozdeeva, A. Tribolet, A. Tronconi, G. Venturi and S.Yu. Vernov, Phys. Rev. D **110** (2024) 104011 [arXiv:2406.19762].

SLOW-ROLL PARAMETERS

In the models with one nonminimally coupled scalar field, there are two sets of the slow-roll parameters:

$$\varepsilon_1 = -\frac{\dot H}{H^2} = \frac{d\ln(H^{-1})}{dN} = -\frac{1}{2}\frac{d\ln(H^2)}{dN}, \qquad \varepsilon_n = \frac{\dot\varepsilon_{n-1}}{H\varepsilon_{n-1}} = \frac{d\ln(\varepsilon_{n-1})}{dN},$$

$$\zeta_1 = \frac{\dot F}{HF} = \frac{d\ln(F)}{dN}, \qquad \zeta_n = \frac{\dot\zeta_{n-1}}{H\zeta_{n-1}} = \frac{d\ln(\zeta_{n-1})}{dN}.$$

We rewrite Eq. (11) as

$$Y = \frac{M_{\rm Pl}H}{\sqrt{F}}\left(1+\frac{1}{2}\zeta_1\right), \qquad (12)$$

and get Eq. (7) in the following form

$$3M_{\rm Pl}^4H^2\left(1+\frac{1}{2}\zeta_1\right)^2 = \frac{AF}{2}\dot\phi^2 + FV_{\rm eff}. \qquad (13)$$

If $V(\phi) > 0$ and $F(\phi) > 0$ for all ϕ in some interval $\phi_1 < \phi < \phi_0$ and $\zeta_1(\phi_0) > -2$, then $\zeta_1(\phi) > -2$ for all ϕ in this interval.

Using slow-roll equations, we get the expressions $H^2(\phi)$, $\chi(\phi)$, and slow-parameters:

$$\zeta_1 = \frac{d\ln(F(\phi))}{d\phi}\,\chi(\phi), \qquad \varepsilon_1 = -\frac{1}{2}\,\frac{d\ln(H^2(\phi))}{d\phi}\,\chi(\phi). \tag{14}$$

The inflationary parameters are

$$r \approx 8\,|2\varepsilon_1 + \zeta_1| = 8\left|\frac{d(\ln(F/H^2))}{d\phi}\chi\right|, \tag{15}$$

$$n_s \approx 1 - 2\varepsilon_1 - \zeta_1 - \frac{2\varepsilon_1\varepsilon_2 + \zeta_1\zeta_2}{2\varepsilon_1 + \zeta_1} = 1 + \frac{r}{8} - \frac{d\ln(r)}{dN}, \tag{16}$$

$$A_s \approx \frac{2H^2}{\pi^2 F\, r}. \tag{17}$$

The inflationary parameters are constrained by the combined analysis of Planck, BICEP/Keck and other observations as follows[1]:

$$A_s = (2.10 \pm 0.03)\times 10^{-9}, \qquad n_s = 0.9654 \pm 0.0040\,, \qquad r < 0.028\,.$$

[1] G. Galloni, N. Bartolo, S. Matarrese, M. Migliaccio, A. Ricciardone and N. Vittorio, JCAP **04** (2023) 062 [arXiv:2208.00188].

THE SIMPLEST APPROXIMATION

In the simplest approximation, one assumes that all slow-roll parameters are negligibly small and get the following system of approximate equations,

$$H^2 \approx \frac{V}{3F}, \tag{18}$$

$$3H\dot{\phi} + V_{,\phi} - 6F_{,\phi}H^2 \approx 0. \tag{19}$$

The expressions of $\chi(\phi)$ and $\zeta_1(\phi)$ are as follows:

$$\chi(\phi) = \frac{\dot{\phi}H}{H^2} \approx 2F_{,\phi} - \frac{FV_{,\phi}}{V}, \tag{20}$$

$$\zeta_1(\phi) = \frac{F_{,\phi}}{F}\chi(\phi) \approx \frac{2F_{,\phi}^2}{F} - \frac{F_{,\phi}V_{,\phi}}{V}. \tag{21}$$

Using Eq. (14), we come to the expression

$$\varepsilon_1(\phi) \approx \frac{FV_{,\phi} - 2F_{,\phi}V}{2V}\left(\frac{V_{,\phi}}{V} - \frac{F_{,\phi}}{F}\right). \tag{22}$$

THE KNOWN SLOW-ROLL APPROXIMATION

This approximation is derived using the transition between Jordan and Einstein frames. One obtains the following slow-roll equations[2]:

$$H^2(\phi) \approx \frac{V}{3F} \tag{23}$$

and

$$\chi(\phi) = \frac{\dot{\phi}H}{H^2} \approx \frac{2F\left(2F_{,\phi}V - FV_{,\phi}\right)}{V\left(2F + 3F_{,\phi}^2\right)}. \tag{24}$$

The parameters $\varepsilon_1(\phi)$ and $\zeta_1(\phi)$ are obtained by Eq. (14):

$$\varepsilon_1(\phi) \approx -\frac{F\left(2F_{,\phi}V - FV_{,\phi}\right)}{V(2F + 3F_{,\phi}^2)}\left(\frac{V_{,\phi}}{V} - \frac{F_{,\phi}}{F}\right), \tag{25}$$

$$\zeta_1(\phi) = \frac{F_{,\phi}}{F}\chi(\phi) \approx \frac{2F_{,\phi}\left(2F_{,\phi}V - FV_{,\phi}\right)}{V\left(2F + 3F_{,\phi}^2\right)}. \tag{26}$$

[2] D.I. Kaiser, Phys. Rev. D **52** (1995) 4295 [arXiv:astro-ph/9408044].

NEW APPROXIMATION I

Neglecting terms proportional to $\dot\phi^2$ and $\ddot\phi$, we reduce Eqs. (2) and (4) as follows

$$3H^2F \approx V - 3F_{,\phi}\dot\phi H, \tag{27}$$

$$3H\dot\phi + V_{,\phi} - 3F_{,\phi}(\dot H + 2H^2) \approx 0. \tag{28}$$

Substituting (27) and (3) to (28) and neglecting again terms proportional to $\dot\phi^2$ and $\ddot\phi$, we find

$$3H\dot\phi \approx -2\left(\frac{FV_{,\phi} - 2F_{,\phi}V}{2F + 3F_{,\phi}^2}\right), \tag{29}$$

Therefore,

$$H^2(\phi) \approx \frac{2FV - F_{,\phi}^2 V + 2FF_{,\phi}V_{,\phi}}{3F(2F + 3F_{,\phi}^2)}. \tag{30}$$

$$\chi(\phi) = \frac{\dot\phi H}{H^2} \approx -\frac{2F(FV_{,\phi} - 2F_{,\phi}V)}{2FV - {F_{,\phi}}^2 V + 2FF_{,\phi}V_{,\phi}}. \tag{31}$$

NEW APPROXIMATIONS II and III

These slow-roll approximations are based on system (7)–(9). Neglecting the proportional to $\dot{\phi}^2$ term in Eq. (7), we get

$$Y^2 \approx \frac{V_{\rm eff}}{3M_{\rm Pl}^2} = \frac{M_{\rm Pl}^2 V}{3F^2}\,. \qquad (32)$$

Differentiating this equation over time and using Eq. (8), we obtain

$$\dot{\phi} \approx -\frac{M_{\rm Pl} V_{{\rm eff},\phi}}{3YA\sqrt{F}} = -\frac{2M_{\rm Pl}\left(V_{,\phi}F - 2VF_{,\phi}\right)}{3Y\sqrt{F}\left(2F + 3F_{,\phi}^2\right)} = \frac{-2\left(V_{,\phi}F - 2VF_{,\phi}\right)}{3H\left(1 + \frac{\zeta_1}{2}\right)\left(2F + 3F_{,\phi}^2\right)}\,.$$

and

$$\chi \approx -\frac{2\left(V_{,\phi}F - 2VF_{,\phi}\right)}{3H^2\left(1 + \frac{\zeta_1}{2}\right)\left(2F + 3F_{,\phi}^2\right)} = -\frac{2F\left(V_{,\phi}F - 2VF_{,\phi}\right)}{V\left(2F + 3F_{,\phi}^2\right)}\left(1 + \frac{F_{,\phi}}{2F}\chi\right)\,.$$

We obtain the first-order differential equation that defines slow-roll dynamic of ϕ:

$$\chi=\frac{d\phi}{dN}=\frac{2F\left(2VF_{,\phi}-V_{,\phi}F\right)}{2VF+V_{,\phi}F_{,\phi}F+VF_{,\phi}^{2}}\,. \qquad (33)$$

Substituting (33) into Eq. (14), we find

$$\zeta_1(\phi)=\frac{2F_{,\phi}(2VF_{,\phi}-V_{,\phi}F)}{2VF+V_{,\phi}FF_{,\phi}+VF_{,\phi}^{2}}. \qquad (34)$$

If we neglect the term proportional to ζ_1^2 in

$$H^2=\frac{FY^2}{M_{\mathrm{Pl}}^2\left(1+\frac{1}{2}\zeta_1\right)^2}\approx\frac{V}{3F\left(1+\zeta_1\right)}, \qquad (35)$$

then we come to ***the Approximation II.***
If we do not neglect any terms in Eq. (35) and use Eq. (33), then we obtain ***the Approximation III***:

$$H^2\approx\frac{V}{3F}\left(1+\frac{1}{2}\zeta_1\right)^{-2}=\frac{\left(2FV+FF_{,\phi}V_{,\phi}+F_{,\phi}^2V\right)^2}{3FV\left(2F+3F_{,\phi}^2\right)^2}, \qquad (36)$$

- ***the Approximation III*** corresponds to the Einstein frame approximation, because all terms in Eq. (7) represent terms from the corresponding Friedmann equation in the Einstein frame.
- The function Y is the Hubble parameter in the Einstein frame, so Eq. (7) differs from the standard form of the Friedmann equation only because there is no transition to the Einstein frame scalar field, so that the kinetic term in Eq. (7) is a non-standard one and the time t is a parametric one in the Einstein frame.
- However, the moments of the end of inflation in the Jordan and Einstein frames do not coincide, because

$$\tilde{\varepsilon}_1 = -\frac{1}{\tilde{H}^2}\frac{d\tilde{H}}{d\tilde{t}} = \varepsilon_1 + \frac{\zeta_1\left(1-\varepsilon_1\right)}{2+\zeta_1} + \frac{2\zeta_1\zeta_2}{\left(2+\zeta_1\right)^2} \approx \varepsilon_1 + \frac{1}{2}\zeta_1\,. \quad (37)$$

This means that despite the above-mentioned correspondence, calculations in this approximation gives results different from obtained directly in the Einstein frame.

THE HIGGS DRIVEN INFLATION

Let us consider the well-known inflationary model [3] with the induced gravity term and the fourth degree monomial potential,

$$F(\phi) = M_{\mathrm{Pl}}^2\left(1+\xi\phi^2\right), \qquad V(\phi) = \frac{\lambda}{4}\phi^4, \tag{38}$$

where ξ and λ are positive constants.
Note that the function $\phi(N)$, a solution of Eq. (6), as well as functions $\chi(\phi)$, $\epsilon_i(\phi)$ and $\zeta_1(\phi)$ calculated in any slow-roll approximation do not depend on λ.
Hence, the values of inflationary parameters n_s and r do not depend on λ and this parameter is important to define the inflationary parameter A_s.
With an additional assumption that ϕ is the Standard model Higgs boson, one chooses $\xi = 17367$ and $\lambda = 0.05$.

[3] B.L. Spokoiny, *Phys. Lett. B* **147** (1984) 39;
A.O. Barvinsky and A.Yu. Kamenshchik, *Phys. Lett. B* **332** (1994) 270;
F.L. Bezrukov and M. Shaposhnikov, *Phys. Lett. B* **659** (2008) 703;
A.O. Barvinsky, A.Yu. Kamenshchik, and A.A. Starobinsky, *JCAP* **11** (2008) 021;
F. Bezrukov, A. Magnin, M. Shaposhnikov, and S. Sibiryakov, *JHEP* **01** (2011) 016;
F. Bezrukov, *Class. Quant. Grav.* **30** (2013) 214001

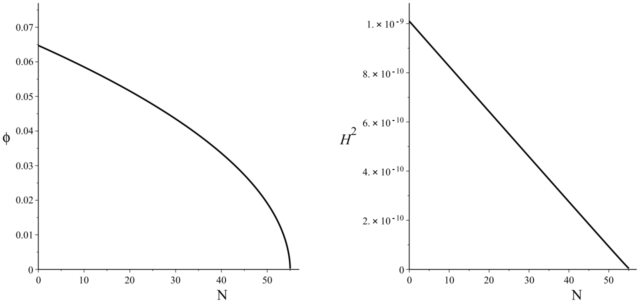

Рис.: The evolution of $\phi(N)$ and $H^2(N)$ in units of $M_{\rm Pl}$ during inflation, obtained by the numerical integration of the system (3).

In Fig. 2, one can see the behavior of the slow-roll parameters ϵ_1 and ζ_1 as functions of ϕ.

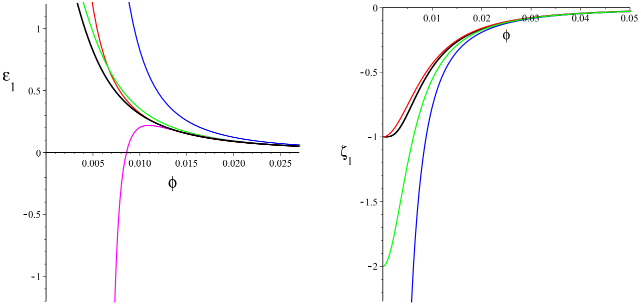

Рис.: The functions $\varepsilon_1(\phi)$ (left) and $\zeta_1(\phi)$ (right). The black lines are the result of the numerical integration of the system (3), the blue curves are obtained in the known approximation, the red curves are obtained in the approximation I, the magenta curve is obtained in the approximation II and the green curves are obtained in the approximation III.

A more interesting and unexpected result is that new approximations I and III give essentially more precise values of the inflationary parameters r and A_s.

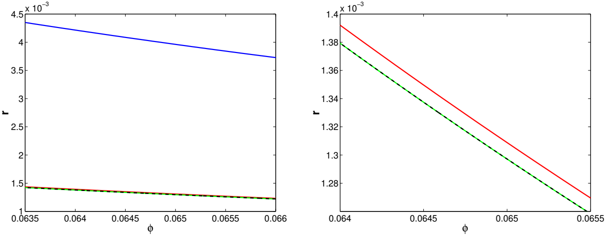

Рис.: The dependence r of the scalar field ϕ for the model with the potential $V(\phi) = 0.0125\phi^4$ and the coupling function $F(\phi) = 1 + 17367\phi^2$. The black lines are the result of the numerical integration of the system (3), the blue curves are obtained in the known approximation, the red curves are obtained in the approximation I and the green curves are obtained in the approximation III.

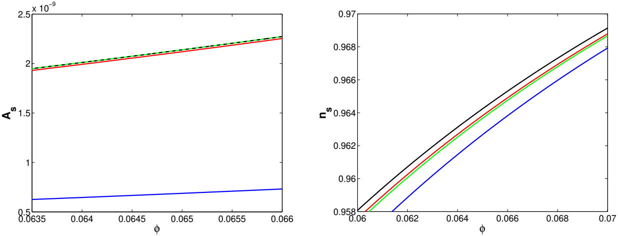

Рис.: The functions $A_s(\phi)$ (left) and $n_s(\phi)$ (right). The black lines are the result of the numerical integration of the system (3), the blue curves are obtained in the known approximation, the red curves are obtained in the approximation I, and the green curves are obtained in the approximation III.

For other values of the model parameters:
$\xi = 1,\ \lambda = 2 \cdot 10^{-10}$
and
$\xi = 2.4 \cdot 10^{9},\ \lambda = 8 \cdot 10^{8}$
we get similar results.

CONCLUSIONS

- We propose new slow-roll approximations for inflationary models with the nonminimal coupling. We find more precise expressions of the slow-roll parameters as functions of the scalar field.
- To check the accuracy of the proposed approximations we consider the Higgs-driven inflationary model. The proposed versions of the slow-roll approximation are not only more accurate at the end of inflation, but also give essentially more precise estimations for the tensor-to-scalar ratio r and of the amplitude of scalar perturbations A_s.
- The standard way to analyze an inflationary model with a nonminimally coupled scalar field starts from the conformal transformation of the metric and construction of the corresponding model in the Einstein frame. This method cannot be used for construction of inflationary models with the Gauss–Bonnet term.

We plan to continue our investigation, adding the Gauss–Bonnet term in the model with the nonminimal coupling and using the results of E.O. Pozdeeva, M.A. Skugoreva, A.V. Toporensky, and S.Yu. Vernov, *JCAP* **09** (2024) 050, arXiv:2403.06147.

CONCLUSIONS

- We propose new slow-roll approximations for inflationary models with the nonminimal coupling. We find more precise expressions of the slow-roll parameters as functions of the scalar field.
- To check the accuracy of the proposed approximations we consider the Higgs-driven inflationary model. The proposed versions of the slow-roll approximation are not only more accurate at the end of inflation, but also give essentially more precise estimations for the tensor-to-scalar ratio r and of the amplitude of scalar perturbations A_s.
- The standard way to analyze an inflationary model with a nonminimally coupled scalar field starts from the conformal transformation of the metric and construction of the corresponding model in the Einstein frame. This method cannot be used for construction of inflationary models with the Gauss–Bonnet term.

We plan to continue our investigation, adding the Gauss–Bonnet term in the model with the nonminimal coupling and using the results of E.O. Pozdeeva, M.A. Skugoreva, A.V. Toporensky, and S.Yu. Vernov, *JCAP* **09** (2024) 050, arXiv:2403.06147.

Спасибо за внимание.