Horndeski theory within time-dependent spherically-symmetric background.

based on papers with S. Mironov and V. Volkova
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Modified gravity
Motivation to study

- Solutions without singularities.
  1. Compact objects
  2. Cosmological solutions
- Other modified gravity solutions (e.g. hairy BH, neutron star)

- Null Energy Condition: $T_{\mu\nu}k^{\mu}k^{\nu} \geq 0$
- Penrose theorem: no singularity $\Rightarrow$ NEC-violation
- Null Convergence Condition: $R_{\mu\nu}k^{\mu}k^{\nu} \geq 0$ (for modified gravity solutions)

Horndeski theory and its generalization

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5 + \mathcal{L}_{\mathcal{BH}}\right),$$

$$\begin{aligned}
\mathcal{L}_2 &= F(\pi, X), \\
\mathcal{L}_3 &= -K(\pi, X)\Box\pi, \\
\mathcal{L}_4 &= G_4(\pi, X)R + G_{4X}(\pi, X)\left[(\Box\pi)^2 - \pi_{;\mu\nu}\pi^{;\mu\nu}\right], \\
\mathcal{L}_5 &= G_5(\pi, X)G^{\mu\nu}\pi_{;\mu\nu} - \frac{1}{6}G_{5X}\left[(\Box\pi)^3 - 3\Box\pi\pi_{;\mu\nu}\pi^{;\mu\nu} + 2\pi_{;\mu\nu}\pi^{;\mu\rho}\pi_{;\rho}{}^{\nu}\right], \\
\mathcal{L}_{\mathcal{BH}} &= F_4(\pi, X)\epsilon^{\mu\nu\rho}{}_{\sigma}\epsilon^{\mu'\nu'\rho'\sigma}\pi_{,\mu}\pi_{,\mu'}\pi_{;\nu\nu'}\pi_{;\rho\rho'} + \\
&\quad + F_5(\pi, X)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\mu'\nu'\rho'\sigma'}\pi_{,\mu}\pi_{,\mu'}\pi_{;\nu\nu'}\pi_{;\rho\rho'}\pi_{;\sigma\sigma'},
\end{aligned}$$

$$X = -\frac{1}{2}g^{\mu\nu}\pi_{,\mu}\pi_{,\nu}$$

Time-dependent spherically symmetric background.

- Background scalar field: $\pi = \pi(r, t)$
- Background metric

$$ds^2 = -A(r,t)\,dt^2 + \frac{dr^2}{B(r,t)} + J^2(r,t)\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right)$$

- Shift-symmetric scalar field (i.e. $\pi(r,t) = q \cdot t + \psi(r)$)
  [T. Kobayashi 1510.07400]

Horndeski theory. G4 subclass.

$$S = \int \mathrm{d}^4x \sqrt{-g} \left(\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 \right),$$

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Perturbations.

- Perturbations

$$\pi = \bar{\pi} + \chi$$

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu},$$

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$$\pi = \bar{\pi} + \chi$$

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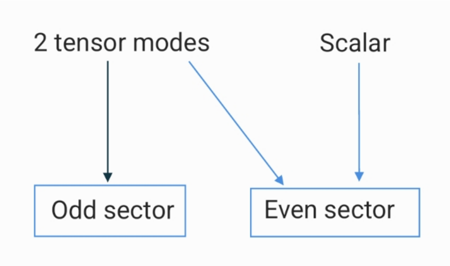
- Regge-Wheeler classification of perturbations
- Odd parity (axial) and even parity (polar) modes (Z_2 rotations).



Even parity sector. Parametrization.

$$
\begin{cases}
h_{tt} = A(t,r)\sum_{\ell,m} H_{0,\ell m}(t,r)Y_{\ell m}(\theta,\varphi), \\
h_{tr} = \sum_{\ell,m} H_{1,\ell m}(t,r)Y_{\ell m}(\theta,\varphi), \\
h_{rr} = \frac{1}{B(t,r)}\sum_{\ell,m} H_{2,\ell m}(t,r)Y_{\ell m}(\theta,\varphi), \\
h_{ta} = \sum_{\ell,m} \beta_{\ell m}(t,r)\partial_a Y_{\ell m}(\theta,\varphi), \\
h_{ra} = \sum_{\ell,m} \alpha_{\ell m}(t,r)\partial_a Y_{\ell m}(\theta,\varphi), \\
h_{ab} = \sum_{\ell,m} K_{\ell m}(t,r)g_{ab}Y_{\ell m}(\theta,\varphi) + \sum_{\ell,m} G_{\ell m}(t,r)\nabla_a\nabla_b Y_{\ell m}(\theta,\varphi).
\end{cases}
$$

$$
\pi(t,r,\theta,\varphi) = \pi(t,r) + \sum_{\ell,m} \chi_{\ell m}(t,r)Y_{\ell m}(\theta,\varphi),
$$

Even parity sector. Gauge transformation.

- $x^\mu \to x^\mu + \xi^\mu$ with ξ^μ parametrized as

$$\xi^\mu = \left(T_{\ell m}(t,r), R_{\ell m}(t,r), \Theta_{\ell m}(t,r)\partial_\theta, \frac{\Theta_{\ell m}(t,r)\partial_\varphi}{\sin^2\theta} \right) Y_{\ell m}(\theta,\varphi)$$

- In use in the static case.

$$\beta = 0, \quad K = 0, \quad G = 0.$$

- **No static scalar field $\pi(r,t)$ limit.**

$$\chi = 0, \quad K = 0, \quad G = 0.$$

- Shift-symmetric solutions:

$$\pi(t,r) = \pi(r) + qt$$

- Stability conditions and propagation speeds do not depend on gauge choice.

High momentum regime in even sector.

- Quadratic action. $v^1 = H_2, \quad v^2 = \alpha$.

$$S^{(2)}_{even} = \int \mathrm{d}t \, \mathrm{d}k \sqrt{\frac{A}{B}} J^2 \left(\mathcal{K}_{ij} \dot{v}^i \dot{v}^j + k \mathcal{Q}_{ij} \dot{v}^i v^j - k^2 \mathcal{G}_{ij} v^i v^j + \ldots \right),$$

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- Dispersion relation:

$$\left(c^2_{r1,2} \mathcal{K}_{ij} - c_{r1,2} (AB)^{-1/2} \mathcal{Q}_{ij} - (AB)^{-1} \mathcal{G}_{ij} \right) |_{Eigenvalues} = 0,$$

$$c^{(\pm)}_{r1} = \sqrt{\frac{B}{A} \frac{\mathcal{J}}{\mathcal{F}} \pm \frac{1}{\mathcal{F}} \sqrt{\mathcal{Z}}} = c^{(\pm)}_{odd}.$$

Stability conditions. G4 subclass. Even sector

$$\left(c_{r1,2}^2\mathcal{K}_{ij} - c_{r1,2}(AB)^{-1/2}\mathcal{Q}_{ij} - (AB)^{-1}\mathcal{G}_{ij}\right)|_{Eigenvalues} = 0,$$

- No-ghost:

$$\mathcal{K}_{11} > 0, \qquad det\ \mathcal{K} > 0.$$

- No radial gradient:

$$\mathcal{G}_{11}^{1,2} > 0, \qquad det\ \mathcal{G}^{1,2} > 0.$$

- Angular gradient.
- Tachyonic (low energies).

Even sector. No-go theorem.

- Static spherically symmetric or cosmological case in Horndeski theory:

  Stability conditions $\Rightarrow$ **no-go theorem.**

  [*V. Rubakov, S. Mironov, M. Libanov*]

Even sector. No-go theorem.

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  Generalizations:
- Additional matter.
- Multi-galileon.

- Several ways to bypass the no-go theorem.
- Absence of the no-go theorem in Beyond Horndeski.

No-go theorem. Spherically symmetric background.

- **Static** case in full Horndeski theory:

  Stability conditions ⇒ no-go theorem.

- **Dynamical** background in cubic subclass:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(R + F(\pi, X) - K(\pi, X)\Box\pi\right)$$

  Stability conditions ⇒ generalized no-go theorem.

No-go theorem. Spherically symmetric background.

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- **Dynamical** background in cubic subclass:

$$S = \int \mathrm{d}^4x\sqrt{-g}\,(R + F(\pi, X) - K(\pi, X)\Box\pi)$$

  Stability conditions ⇒ generalized no-go theorem.

$$\mathcal{K}^{00}\omega^2 = \mathcal{K}^{rr}{k_r}^2 + \mathcal{K}^{\Omega}{k_\phi}^2 + \mathcal{K}^{tr}\omega k_r.$$

$$\begin{cases} \mathcal{K}^{00} > 0 \\ \mathcal{K}^{\Omega} \geq 0 \\ \mathcal{K}^{rr} \geq -\frac{(\mathcal{K}^{tr})^2}{4\mathcal{K}^{00}} \end{cases}$$

Sufficient conditions of the generalized no-go theorem.

- Coordinate transformation $\Rightarrow$ π is a function of one variable.
- $\exists \gamma(\lambda) \in (r, t)$: at any point of γ, its tangent vector $\xi^\mu = \frac{\partial x^\mu}{\partial \lambda}$ satisfies $\xi_\mu = \partial_\mu \pi(r, t)$.
- In the region around the curve γ:
  1. The stability conditions are satisfied.
  2. The field equations are satisfied.
  3. The curve γ is either timelike or spacelike.
  4. γ does not contain zeros and singularities of the background functions.

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  3. The curve γ is either timelike or spacelike.
  4. γ does not contain zeros and singularities of the background functions.

As a result:

- γ is spacelike, $\pi \mid_\gamma = \pi(r')$ $\Rightarrow$ Static no-go th.
- The case reduces to either the static or cosmological no-go theorem.

Generalized no-go in the cubic subclass.

- Sufficient conditions **only** in the region around the γ.
- Now the no-go theorem applies to solutions with special points.

Odd parity sector. Horndeski theory + F4

$$S = \int \mathrm{d}^4 x \sqrt{-g} \left(\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 \right),$$

$$\mathcal{L}_2 = F(\pi, X),$$

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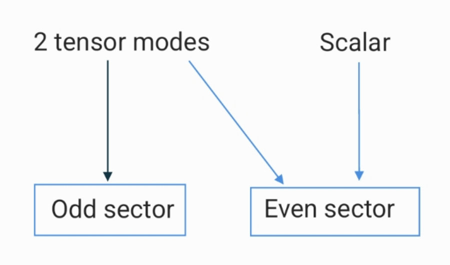

Odd parity sector. Horndeski theory + F4

- Quadratic action

$$\mathcal{L}^{(2)}_{odd} = \sqrt{\frac{B}{A}} J^2 \frac{\ell(\ell+1)}{2(\ell-1)(\ell+2)} \cdot \left[\frac{1}{A} \frac{\mathcal{F}\mathcal{H}^2}{\mathcal{Z}} \dot{Q}^2 - \frac{B \cdot \mathcal{K}\mathcal{H}^2}{\mathcal{Z}} (Q')^2 \right.$$
$$\left. +2 \frac{B}{A} \frac{\mathcal{J}\mathcal{H}^2}{\mathcal{Z}} Q' \dot{Q} - \frac{\ell(\ell+1)}{J^2} \cdot \mathcal{H} Q^2 - V(r) Q^2 \right] ,$$

- Stability analysis

Odd parity sector. Horndeski theory + F4

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- Stability analysis
- Absence of the no-go theorem in the odd parity sector.

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- Stability analysis
- Absence of the no-go theorem in the odd parity sector.
- Propagation speeds

$$c_r^{(\pm)} = \sqrt{\frac{B}{A}} \frac{\mathcal{J}}{\mathcal{F}} \pm \frac{1}{\mathcal{F}} \sqrt{\mathcal{Z}} \leq 1, \qquad c_\theta^2 = \frac{\mathcal{Z}}{\mathcal{F}\mathcal{H}} \leq 1$$

Odd parity. Additional restrictions.

- After GW170817: $|c_{GW}/c_\gamma - 1| \leq 5 \times 10^{-16}$
- Restrictions for the propagation speeds

$$c_r^{(\pm)} = \sqrt{\frac{B}{A}}\frac{\mathcal{J}}{\mathcal{F}} \pm \frac{1}{\mathcal{F}}\sqrt{\mathcal{Z}} = 1, \qquad c_\theta^2 = \frac{\mathcal{Z}}{\mathcal{F}\mathcal{H}} = 1,$$

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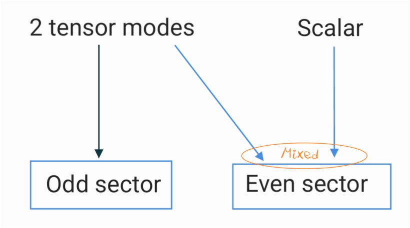
- The only viable subclass of BH theory
  1. $G_5(\pi, X) = 0$
  2. $F_4(\pi, X) = \frac{G_{4X}(\pi,X)}{2X}$
  3. Arbitrary $G_4(\pi, X)$

Speeds of graviton.



Odd | **Even**

1. Radial speeds

$$c_{r,Q}^{(\pm)} = c_{r,\mathcal{V}}^{(\pm)} = \sqrt{\frac{B}{A}\frac{\mathcal{J}}{\mathcal{F}} \pm \frac{1}{\mathcal{F}}\sqrt{\mathcal{Z}}} = 1. \quad \Big| \quad c_{r,g}^{(\pm)} = c_{r,\mathcal{V}}^{(\pm)} = \sqrt{\frac{B}{A}\frac{\mathcal{J}}{\mathcal{F}} \pm \frac{1}{\mathcal{F}}\sqrt{\mathcal{Z}}} = 1.$$

2. Angular speeds

$$c_{\theta,Q}^2 = c_{\theta,\mathcal{V}}^2 = \frac{\mathcal{Z}}{\mathcal{F}\mathcal{H}} = 1. \quad \Big| \quad \text{mixed}$$

Kaluza-Klein compactification.

- Compactification $R^5 \longrightarrow R^4 \times S^1$
- Kaluza-Klein metric, 5D theory

$$g_{AB} = \begin{pmatrix} g_{\mu\nu} + \phi^2 A_\mu A_\nu & \phi^2 A_\mu \\ \phi^2 A_\nu & \phi^2 \end{pmatrix}$$

- Horndeski action in 5D theory

$$\mathcal{S}_5 = \int \mathrm{d}^5 x \sqrt{-g_{(5)}} \Big(G_2(\pi, X) + G_3(\pi, X) \Box\pi$$

$$+ G_4 R_{(5)} + G_{4,X} \left((\Box\pi)^2 - (\nabla_A \nabla_B \pi)^2 \right) + G_5(\pi) G^{AB} \nabla_A \nabla_B \pi \Big)$$

- Cylindrical conditions.

KK compactification of Horndeski theory.

$$\phi(\mathcal{L}_2+\mathcal{L}_3+\mathcal{L}_4)+\mathcal{L}_{4A_\mu}+\mathcal{L}_{4\phi}=$$
$$\int d^4x\,\sqrt{-g}\,\phi\,\Big[G_2(\pi,\,X)+G_3(\pi,\,X)\,\Box\pi+G_4(\pi,\,X)\,\left(R-\frac{1}{4}\phi^2\,F^2-2\frac{\Box\phi}{\phi}\right)$$
$$+G_{4,X}(\pi,\,X)\left((\Box\pi)^2-(\nabla_\mu\nabla_\nu\pi)^2+2\,\frac{1}{\phi}\,\nabla_\mu\phi\nabla^\mu\pi\,\Box\pi-\frac{1}{2}\phi^2\,F_\mu{}^\sigma\,F_{\nu\sigma}\,\nabla^\mu\pi\,\nabla^\nu\pi\right)\Big]$$

$$\phi\mathcal{L}_5+\mathcal{L}_{5A_\mu}+\mathcal{L}_{5\phi}=\int d^4x\,\sqrt{-g}\,\phi\,G_5(\pi)\Big[\left(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}\,R\right)\nabla_\mu\nabla_\nu\pi$$
$$-\frac{1}{2\,\phi}R\,\nabla_\mu\phi\,\nabla^\mu\pi+\frac{1}{\phi}\,(\Box\phi\,\Box\pi-\nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\pi)+\frac{1}{2}\phi^2\,F_{\mu\nu}\,\nabla_\sigma F^{\nu\sigma}\,\nabla^\mu\pi$$
$$+\frac{1}{8}\phi\,F^{\mu\nu}\,F^{\sigma\rho}\left(3\,g_{\nu\rho}(-4\,g_{\lambda\mu}\,g_{\beta\sigma}+g_{\lambda\beta}\,g_{\mu\sigma})\,\nabla^\lambda\pi\nabla^\beta\phi+\phi\,g_{\sigma\mu}\,(-4\,\nabla_\nu\nabla_\rho\pi+g_{\rho\nu}\Box\pi)\right)\Big]$$

KK compactification of Horndeski theory.

$$R^5 \quad \longrightarrow \quad R^4 \times S^1$$

- Generalized Galileons $\longrightarrow$ Generalized Galileons
- 2nd derivatives in the action $\longrightarrow$ 2nd derivatives in the action
- no higher derivatives in EOMs $\longrightarrow$ no higher derivatives in EOMs
- Metric + scalar $\longrightarrow$ Metric + vector + scalar + scalar [U(1) gauge]

Time-dependent spherically symmetric background.

- Background metric

$$ds^2 = -A(r,t)\,dt^2 + \frac{dr^2}{B(r,t)} + J^2(r,t)\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right)$$

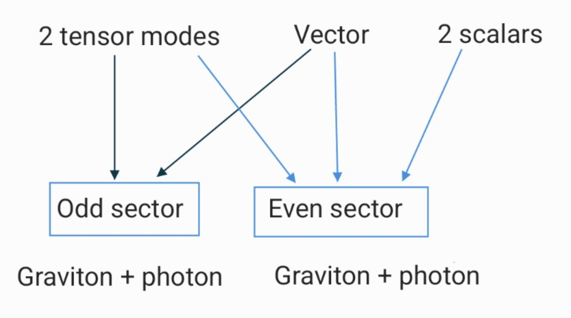
- Background fields

$$\pi = \pi(r,t), \qquad \phi = \phi(t,r), \qquad A_\mu = (A_0(t,r),\ A_1(t,r),\ 0,\ 0).$$

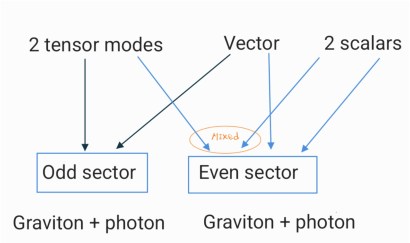
Classification of perturbations.

$$\pi = \bar{\pi} + \chi, \qquad g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu},$$

$$\phi = \bar{\phi} + \delta\phi, \qquad A_\mu = \bar{A}_\mu + \delta A_\mu.$$



Speeds of graviton and modified photon



Odd		**Even**

1. Radial speeds

$$c_{r,Q}^{(\pm)} = c_{r,\mathcal{V}}^{(\pm)} = \sqrt{\frac{B}{A}\frac{\mathcal{J}}{\mathcal{F}} \pm \frac{1}{\mathcal{F}}\sqrt{\mathcal{Z}}} \neq 1. \qquad \Big| \qquad c_{r,g}^{(\pm)} = c_{r,\mathcal{V}}^{(\pm)} = \sqrt{\frac{B}{A}\frac{\mathcal{J}}{\mathcal{F}} \pm \frac{1}{\mathcal{F}}\sqrt{\mathcal{Z}}} \neq 1.$$

2. Angular speeds

$$c_{\theta,Q}^{2} = c_{\theta,\mathcal{V}}^{2} = \frac{\mathcal{Z}}{\mathcal{F}\mathcal{H}} \neq 1. \qquad \Big| \qquad \text{mixed}$$

(The notations were saved from the non-compactified theory)

Conclusion and outlook

- General time-dependent spherically symmetric background within Horndeski theory.
- Stability conditions.
- Speeds of gravity waves.
- Generalized no-go theorem in the cubic subclass.
- KK compactification of Horndeski theory.
- $c_{GW} = c_{\mathcal{V}}$