A cosmological bounce in the theory of gravity with non-minimal derivative coupling

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Motivation

- GR has successfully been exploited for a long time to describe celestial motion in Solar system, a bending of light rays, gravitational waves, the universe expansion (ΛCDM model)
- GR is unable to solve the number already existing problems and appearing new ones
  - cosmological and black hole singularities
  - dark energy (accelerating expansion of the Universe)
  - initial inflation
  - large scale structure of the universe
  - dark matter evidence
  - cosmological constant problem
  - etc...
- These amazing discoveries have set new serious challenges before theoretical cosmology faced the necessity of radical *modification* or *extension* of General Relativity

Scalar-tensor theories

$$S = \int d^4x\sqrt{-g}\left[F(\phi)R - Z(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2U(\phi)\right] + S_m\left[\psi_m, g_{\mu\nu}\right]$$

- generalizations of the Brans-Dicke theories
- the scalar field is
  - minimally coupled with ordinary matter (physical or Jordan frame)
  - non-minimally coupled with the scalar curvature by the term $F(\phi)R$

Notice: Non-minimal coupling of the scalar field with the scalar curvature is provided by the terms $F(\phi)R$

In 1974, *Gregory Walter Horndeski* derived the action of the most general scalar-tensor theories with second-order equations of motion [G.Horndeski, *Second-Order Scalar-Tensor Field Equations in a Four-Dimensional Space*, IJTP **10**, 363 (1974)]

Horndeski Lagrangian:[1]

$$L_{\rm H} = \sqrt{-g}\,(\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5)$$

$$\begin{aligned}
\mathcal{L}_2 &= G_2(\phi, X)\ , \\
\mathcal{L}_3 &= G_3(\phi, X)\,\Box\phi\ , \\
\mathcal{L}_4 &= G_4(\phi, X)R - 2G_{4,X}(\phi, X)(\Box\phi^2 - \phi^{\mu\nu}\phi_{\mu\nu})\ , \\
\mathcal{L}_5 &= G_5(\phi, X)G_{\mu\nu}\phi^{\mu\nu} + \tfrac{1}{3}G_{5,X}(\phi, X)(\Box\phi^3 - 3\,\Box\phi\,\phi_{\mu\nu}\phi^{\mu\nu} + 2\,\phi_{\mu\nu}\phi^{\mu\sigma}\phi^{\nu}{}_{\sigma})\ ,
\end{aligned}$$

$G_a(\phi, X)$ are four arbitrary functions, and $X = -\frac{1}{2}(\nabla\phi)^2$

Notice: Non-minimal coupling of the scalar field with curvature is provided by two terms, $G_4(\phi, X)R$ and $G_5(\phi, X)G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$

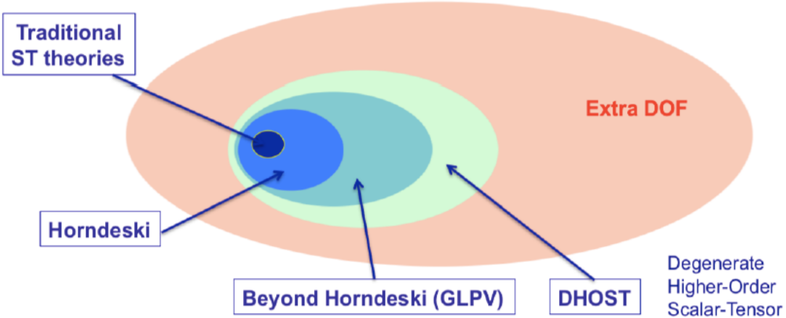
[1] T. Kobayashi, M. Yamaguchi, J. Yokoyama, Prog. Theor. Phys. **126**, 511 (2011).

Subclasses of the Horndeski theory

$$\mathcal{L}_H = \mathcal{L}\{G_2, G_3, G_4, G_5\}$$

- Hilbert-Einstein action (GR):
  $G_4(\phi, X) = \frac{1}{2}M_{Pl}^2 \quad \rightarrow \quad \mathcal{L}_H \sim \frac{1}{2}M_{Pl}^2 R$
- Nonminimal coupling: $G_4(\phi, X) = f(\phi) \quad \rightarrow \quad \mathcal{L}_H \sim f(\phi)R$
- GR with a scalar field: $G_2(\phi, X) = \epsilon X - V(\phi)$
- k-essence: $G_2 = K(\phi, X)$
- Kinetic gravity braiding (KGB):
  $G_3 = B(\phi, X) \quad \rightarrow \quad \mathcal{L}_H \sim B(\phi, X)\Box\phi$
- Nonminimal kinetic coupling:
  $G_5(\phi, X) = \eta\phi \quad \rightarrow \quad \mathcal{L}_H \sim \eta G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$
- Fab Four, Gallileons, etc.

Extended scalar-tensor theories



Landscape of scalar-tensor theories
D. Langlois, Dark energy and modified gravity in degenerate higher-order scalar-tensor (DHOST) theories: A review Int. J. Mod. Phys. D 28 (2019), no. 05 1942006

DHOST theories

$$S = \int d^4x\,\sqrt{-g}\Big[F_{(2)}(\phi,X)R + P(\phi,X) + Q(\phi,X)\Box\phi$$

$$+F_{(3)}(\phi,X)\,G_{\mu\nu}\phi^{\mu\nu} + \sum_{a=1}^{5} A_a(\phi,X)L_a^{(2)} + \sum_{a=1}^{10} B_a(\phi,X)L_a^{(3)}\Big]$$

$$L_1^{(2)} = \phi_{\mu\nu}\phi^{\mu\nu}\,, \qquad L_2^{(2)} = (\Box\phi)^2\,, \qquad L_3^{(2)} = (\Box\phi)\phi^\mu\phi_{\mu\nu}\phi^\nu\,,$$
$$L_4^{(2)} = \phi^\mu\phi_{\mu\rho}\phi^{\rho\nu}\phi_\nu\,, \qquad L_5^{(2)} = (\phi^\mu\phi_{\mu\nu}\phi^\nu)^2\,.$$

$$L_1^{(3)} = (\Box\phi)^3\,, \quad L_2^{(3)} = (\Box\phi)\,\phi_{\mu\nu}\phi^{\mu\nu}\,, \quad L_3^{(3)} = \phi_{\mu\nu}\phi^{\nu\rho}\phi^\mu_\rho\,,$$
$$L_4^{(3)} = (\Box\phi)^2\,\phi_\mu\phi^{\mu\nu}\phi_\nu\,, \quad L_5^{(3)} = \Box\phi\,\phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^\rho\,, \quad L_6^{(3)} = \phi_{\mu\nu}\phi^{\mu\nu}\phi_\rho\phi^{\rho\sigma}\phi_\sigma\,,$$
$$L_7^{(3)} = \phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^{\rho\sigma}\phi_\sigma\,, \quad L_8^{(3)} = \phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^\rho\,\phi_\sigma\phi^{\sigma\lambda}\phi_\lambda\,,$$
$$L_9^{(3)} = \Box\phi\,(\phi_\mu\phi^{\mu\nu}\phi_\nu)^2\,, \quad L_{10}^{(3)} = (\phi_\mu\phi^{\mu\nu}\phi_\nu)^3\,.$$

Notice: Non-minimal coupling of the scalar field with curvature is provided by two terms, $F_{(2)}(\phi,X)R$ and $F_{(3)}(\phi,X)G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$

Non-minimal coupling of the scalar field with curvature

Notice: There are only two qualitatively different terms describing non-minimal coupling of the scalar field with curvature: $M(\phi, X)R$ and $N(\phi, X)G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$.

- $M(\phi, X)R$ — Brans-Dicke-like theories
- $N(\phi, X)G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$ — theories with non-minimal derivative coupling

Theory with nonminimal derivative coupling. I

Focusing on non-minimal derivative coupling, we have

Action: $S = S^{(g)} + S^{(m)}$

$S^{(m)}$ — *the action for ordinary matter fields*

$$S^{(g)} = \frac{1}{2}\int d^4x\sqrt{-g}\left[M_{\rm Pl}^2(R-2\Lambda) - (\varepsilon\, g_{\mu\nu} + \eta\, G_{\mu\nu})\nabla^\mu\phi\nabla^\nu\phi - 2V(\phi)\right]$$

Λ — *cosmological constant*

$\varepsilon = 1$ *(ordinary scalar field)*;
$\varepsilon = -1$ *(phantom scalar field)*;
$\varepsilon = 0$ *(no standard kinetic term)*

η — *dimensional coupling parameter*; $[\eta] = (\mathit{length})^2 \;\rightarrow\; \eta = \pm\ell^2$

ℓ — *characteristic scale of non-minimal coupling*

Field equations:

$$G_{\mu\nu} = -g_{\mu\nu}\Lambda + 8\pi\left[T^{(m)}_{\mu\nu} + T^{(\phi)}_{\mu\nu} + \eta\,\Theta_{\mu\nu}\right]$$

$$[\varepsilon g^{\mu\nu} + \eta G^{\mu\nu}]\nabla_\mu\nabla_\nu\phi = V'_\phi$$

$$T^{(\phi)}_{\mu\nu} = \varepsilon\left[\nabla_\mu\phi\nabla_\nu\phi - \tfrac{1}{2}g_{\mu\nu}(\nabla\phi)^2\right] - g_{\mu\nu}V(\phi),$$

$$\Theta_{\mu\nu} = -\tfrac{1}{2}\nabla_\mu\phi\,\nabla_\nu\phi\,R + 2\nabla_\alpha\phi\,\nabla_{(\mu}\phi R^\alpha_{\nu)} - \tfrac{1}{2}(\nabla\phi)^2 G_{\mu\nu} + \nabla^\alpha\phi\,\nabla^\beta\phi\,R_{\mu\alpha\nu\beta}$$
$$+\nabla_\mu\nabla^\alpha\phi\,\nabla_\nu\nabla_\alpha\phi - \nabla_\mu\nabla_\nu\phi\,\Box\phi + g_{\mu\nu}\big[-\tfrac{1}{2}\nabla^\alpha\nabla^\beta\phi\,\nabla_\alpha\nabla_\beta\phi + \tfrac{1}{2}(\Box\phi)^2$$
$$-\nabla_\alpha\phi\,\nabla_\beta\phi\,R^{\alpha\beta}\big]$$

$$T^{(m)}_{\mu\nu} = (\rho + p)u_\mu u_\mu + p g_{\mu\nu}$$

Notice: *The field equations are of second order!*

Ansatz: $V \equiv 0$ (no potential), $\varepsilon = +1$ (ordinary scalar)
$\phi = \phi(t)$, $T^{(m)}_{\mu\nu} = diag(\rho(t), p(t), p(t), p(t))$, and
the FLRW metric

$$ds^2 = -dt^2 + \mathrm{a}^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right]$$

$k = 0, \pm 1$, $\mathrm{a}(t)$ *cosmological factor*, $H(t) = \dot{\mathrm{a}}(t)/\mathrm{a}(t)$ *Hubble parameter*

Gravitational equations:

$$3\left(H^2 + \frac{k}{\mathrm{a}^2}\right) = \Lambda + 8\pi\rho + 4\pi\psi^2\left(1 - 9\eta\left(H^2 + \frac{k}{3\mathrm{a}^2}\right)\right),$$

$$2\dot{H} + 3H^2 + \frac{k}{\mathrm{a}^2} = \Lambda - 8\pi p - 4\pi\psi^2\left[1 + 2\eta\left(\dot{H} + \frac{3}{2}H^2 - \frac{k}{\mathrm{a}^2} + 2H\frac{\dot{\psi}}{\psi}\right)\right]$$

The scalar field equations:

$$\mathrm{a}^3\psi\left(1 - 3\eta\left(H^2 + \frac{k}{\mathrm{a}^2}\right)\right) = Q = const$$

where $\psi = \dot{\phi}$

Material content is a mixture of radiation and non-relativistic component:

$$\rho = \rho_m + \rho_r = \rho_{m0}\left(\frac{\mathrm{a}_0}{\mathrm{a}}\right)^3 + \rho_{r0}\left(\frac{\mathrm{a}_0}{\mathrm{a}}\right)^4$$

Introducing the dimensionless scales factor a, Hubble parameter h, and coupling parameter ζ:

$$a = \frac{\mathrm{a}}{\mathrm{a}_0}, \quad h = \frac{H}{H_0}, \quad \zeta = \eta H_0^2,$$

and the dimensionless density parameters:

$$\Omega_0 = \frac{\Lambda}{3H_0^2}, \quad \Omega_2 = \frac{k}{\mathrm{a}_0^2 H_0^2}, \quad \Omega_3 = \frac{\rho_{m0}}{\rho_{cr}}, \quad \Omega_4 = \frac{\rho_{r0}}{\rho_{cr}}, \quad \Omega_6 = \frac{4\pi Q^2}{3\mathrm{a}_0^6 H_0^2},$$

where $\rho_{cr} = 3H_0^2/8\pi$ is the critical density, one has

Modified Friedmann equation

$$h^2 = \Omega_0 - \frac{\Omega_2}{a^2} + \frac{\Omega_3}{a^3} + \frac{\Omega_4}{a^4} + \frac{\Omega_6\left(1 - 3\zeta(3h^2 + \frac{\Omega_2}{a^2})\right)}{a^6\left(1 - 3\zeta(h^2 + \frac{\Omega_2}{a^2})\right)^2}$$

Modified Friedmann equation (Master equation). II

Modified Friedmann equation

$$h^2 = \Omega_0 - \frac{\Omega_2}{a^2} + \frac{\Omega_3}{a^3} + \frac{\Omega_4}{a^4} + \frac{\Omega_6\left(1 - 3\zeta(3h^2 + \frac{\Omega_2}{a^2})\right)}{a^6\left(1 - 3\zeta(h^2 + \frac{\Omega_2}{a^2})\right)^2}$$

- Assuming $\Lambda \geq 0$, one has $\Omega_0 \geq 0$
- $\Omega_2 = k/a_0^2 H_0^2$, hence
  $\Omega_2 = 0$, $\Omega_2 < 0$, $\Omega_2 > 0$ if $k = 0, -1, +1$, respectively
- $\zeta = \eta H_0^2 = \pm\left(\ell/\ell_H\right)^2$, where $\ell_H = 1/H_0$, hence
  ζ is proportional to the square of ratio of two characteristic scales,
  hence $\zeta \ll 1$???
- In case $\Omega_6 = 0$ (no scalar with non-minimal derivative coupling) one has the standard master equation of ΛCDM cosmological model
- In case $\Omega_6 \neq 0$ but $\zeta = 0$ (no non-minimal derivative coupling) one has a cosmological model with an ordinary scalar field

Modified Friedmann equation (Master equation). III

Denoting $y = h^2$ one can rewrite the master equation as a cubic in y algebraic equation

$$c_3 y^3 + c_2(a) y^2 + c_1(a) y + c_0(a) = 0$$

with the coefficients

$$\begin{aligned}
c_3 =& 9\zeta^2 \\
c_2 =& -6\zeta\left(1 - \frac{3\zeta\Omega_2}{a^2}\right) - 9\zeta^2\left(\Omega_0 - \frac{\Omega_2}{a^2} + \frac{\Omega_3}{a^3} + \frac{\Omega_4}{a^4}\right), \\
c_1 =& \left(1 - \frac{3\zeta\Omega_2}{a^2}\right)^2 + 6\zeta\left(1 - \frac{3\zeta\Omega_2}{a^2}\right)\left(\Omega_0 - \frac{\Omega_2}{a^2} + \frac{\Omega_3}{a^3} + \frac{\Omega_4}{a^4}\right) + \frac{9\zeta\Omega_6}{a^6}, \\
c_0 =& -\left(1 - \frac{3\zeta\Omega_2}{a^2}\right)^2\left(\Omega_0 - \frac{\Omega_2}{a^2} + \frac{\Omega_3}{a^3} + \frac{\Omega_4}{a^4}\right) - \left(1 - \frac{3\zeta\Omega_2}{a^2}\right)\frac{\Omega_6}{a^6}.
\end{aligned}$$

Notice: Roots $h = h(a)$ of the cubic polynomial (14) define a global cosmological behavior as follows

$$\int_{a=1}^{a} \frac{d\tilde{a}}{\tilde{a} h(\tilde{a})} = H_0(t - t_0).$$

A turning point in the Universe evolution may occur at a moment $t = t_*$, when the scale factor $\mathrm{a}(t)$ reaches its extremal, either maximal or minimal value, $a(t_*) = a_*$. Correspondingly, $y(a_*) = h^2(a_*) = 0$.

The polynomial $P(a, y) = c_3 y^3 + c_2(a) y^2 + c_1(a) y + c_0(a)$ has a root $y(a_*) = 0$ if and only if $c_0(a_*) = 0$,and hence we obtain two separate algebraic conditions for a_*:

$$\left(1 - \frac{3\zeta\Omega_2}{a_*^2}\right)\left(\Omega_0 - \frac{\Omega_2}{a_*^2} + \frac{\Omega_3}{a_*^3} + \frac{\Omega_4}{a_*^4}\right) + \frac{\Omega_6}{a_*^6} = 0, \tag{1}$$

$$\left(1 - \frac{3\zeta\Omega_2}{a_*^2}\right) = 0. \tag{2}$$

NOTICE: The conditions (1) and (2) have NO solutions in case $\Omega_2 \leq 0$. Therefore, in cosmological models with negative or zero spatial curvature there are no turning points.

Turning points and bounces: $\Omega_2 > 0$ (positive spatial curvature)

Condition 1: $$\left(1 - \frac{3\zeta\Omega_2}{a_*^2}\right)\left(\Omega_0 - \frac{\Omega_2}{a_*^2} + \frac{\Omega_3}{a_*^3} + \frac{\Omega_4}{a_*^4}\right) + \frac{\Omega_6}{a_*^6} = 0$$

In the simplest case: $\Omega_0 = \Omega_3 = \Omega_4 = 0, \zeta = 0$, one has

$$a_*^2 = \sqrt{\Omega_6/\Omega_2} = \sqrt{(1+\Omega_2)/\Omega_2}.$$

Supposing $\Omega_2 \ll 1$, we get $a_*^2 = a_{max}^2 \approx 1/\Omega_2^{1/2} \gg 1$

Therefore, the Universe's expansion is stopped when the scale factor achieves its maximal value a_{max} and then replaced by contraction.

This is a turning point!

Thus, a root (if exists) of the Condition 1 gives a maximal value $a_* = a_{max}(\Omega_0, \Omega_2, \Omega_3, \Omega_4, \zeta)$ which does generally depend on *all* parameters of the model.

Turning points and bounces: $\Omega_2 > 0$ (positive spatial curvature)

Condition 2: $$1 - \frac{3\zeta\Omega_2}{a_*^2} = 0 \quad \longrightarrow \quad a_*^2 = 3\zeta\Omega_2$$

Since $\Omega_2 \ll 1$ and $\zeta \ll 1$, we get $a_*^2 = a_{min}^2 \ll 1$

Therefore, the Universe's contraction is stopped when the scale factor achieves its minimal value $a_{min} = (3\zeta\Omega_2)^{1/2}$.

NOTICE:

- The value $a_{min} = (3\zeta\Omega_2)^{1/2}$ depends ONLY on the product $\zeta\Omega_2$, and does NOT depend on Ω_0, Ω_3, Ω_4!
- Following [a], we may say that the cosmological constant and material substance are *screened* at the early stage and makes no contribution to the universe evolution.

[a] A. A. Starobinsky, S. V. Sushkov, and M. S. Volkov, *The screening Horndeski cosmologies*, *JCAP* **1606** (2016), no. 06 007

Let us consider an asymptotic behavior of h near $a = a_* \equiv (3\zeta\Omega_2)^{1/2}$:

$$9\zeta h^2 \approx 2\frac{\Delta a}{a_*} - 3\left(\frac{\Delta a}{a_*}\right)^2 + 4\left(\frac{\Delta a}{a_*}\right)^3 + \dots,$$

where $\Delta a = a - a_*$.
Integrating, we obtain

$$a(\tau) = a_{min}\left(1 + \frac{\Delta\tau^2}{18\zeta}\right) + O(\Delta\tau^4),$$
$$h(\tau) = \frac{\Delta\tau}{9\zeta} + O(\Delta\tau^3),$$

where $a_{min} = a_*$, $\Delta\tau = \tau - \tau_*$, and τ_* is a constant of integration.

Evidently: $a(\tau) \to a_{min}$ and $h(\tau) \to 0$ as $\Delta\tau \to 0$, i.e. $\tau \to \tau_*$.

NOTICE: The spacetime geometry is regular when approaching to the "bounce" a_{min}!

Bounce solution

Is the point $a_*^2 = a_{min}^2 = 3\zeta\Omega_2$ a bounce?

Scalar field equation:

$$\phi' = \frac{q}{a^3\left(1 - 3\zeta\left(h^2 + \frac{\Omega_2}{a^2}\right)\right)}$$

Asymptotics:

$$\phi' \approx \frac{27\zeta q}{2a_{min}^3 \Delta\tau^2}.$$

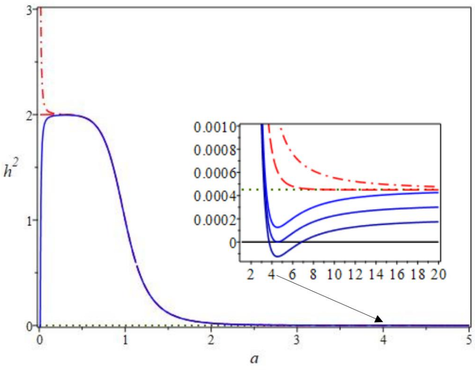
Thus, one has $\phi' \propto 1/\Delta\tau^2 \to \infty$ as $\Delta\tau \to 0$, i.e. $\tau \to \tau_*$.

NOTICE: One has a singular behavior of the scalar field when approaching to the "bounce" a_{min}.

A 'singular' bounce!

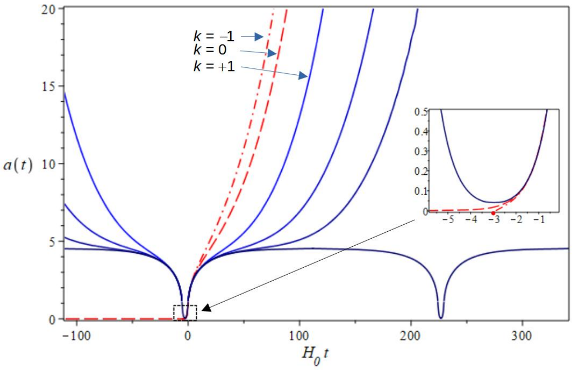
Examples of numerical solutions. I.
The case $\zeta \neq 0$ and $\Omega_0 = \Omega_3 = \Omega_4 = 0$

Plots of h^2 versus a



Examples of numerical solutions. II.
The case $\zeta \neq 0$ and $\Omega_0 = \Omega_3 = \Omega_4 = 0$

Plots of a versus t



Concluding remarks

- We have explored bounce scenarios in the framework of homogeneous and isotropic cosmological models with arbitrary spatial curvature in the theory of gravity with non-minimal derivative coupling.
- In general, the model depends on five independent dimensionless parameters: the coupling parameter ζ, and density parameters Ω_0, Ω_2, Ω_3, Ω_4.
- A bounce cosmological scenario is most general for the homogeneous and isotropic cosmological model with positive spatial curvature ($\Omega_2 > 0$, $k = +1$).
- Near the bounce one has $a(\tau) \approx a_{min}\left(1 + \Delta\tau^2/18\zeta\right) \to a_{min}$ and $h(\tau) \approx \Delta\tau/9\zeta \to 0$ as $\Delta\tau \to 0$. Therefore, the spacetime geometry is regular when approaching to the bounce.
- However, the scalar field diverges near the bounce as follows: $\phi' \propto 1/\Delta\tau^2 \to \infty$ as $\Delta\tau \to 0$.
- Therefore, we can term this scenario as a **'singular' bounce**.

THANKS FOR YOUR ATTENTION!