Factorization of the Hilbert space of eternal black holes in general relativity

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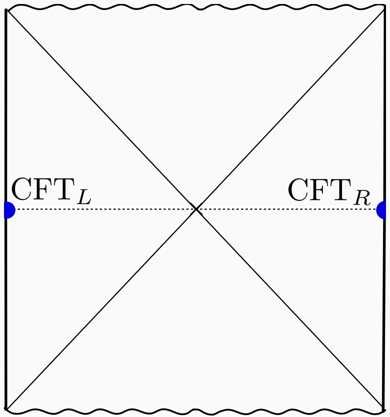
based on 2410.00091 with Vijay Balasubramanian, Ben Craps, Juan Hernandez and Maria Knysh

Hilbert space factorization puzzle in AdS/CFT



Bulk: theory of gravity in AdS_{d+1} with Hilbert space $\mathcal{H}_{\text{bulk}}$.
Boundary: two copies of CFT with Hilbert space $\mathcal{H}_L \otimes \mathcal{H}_R$.
The puzzle: how to see from the bulk view that

$$\mathcal{H}_{\text{bulk}} \simeq \mathcal{H}_L \otimes \mathcal{H}_R\,?$$

Main idea

Let us pick operators $k_{L,R} = e^{-\beta_{L,R} H}$ in the $\mathrm{CFT}_{L,R}$. We denote their bulk duals as $K_{L,R}$ (constructed according to HKLL). Then the factorization holds if and only if[1]

$$Z_{\text{bulk}} = \mathrm{Tr}_{\mathcal{H}_{\text{bulk}}} K_L K_R = \mathrm{Tr}_{\mathcal{H}_L} k_L \, \mathrm{Tr}_{\mathcal{H}_R} k_R \, .$$

How to compute this trace?

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How to compute this trace?

- Construct a family of κ special semiclassical bulk states in the microcanonical ensemble (Balasubramanian et. al., 2212.02447)

$$\mathcal{H}^E_{\mathrm{bulk}}(\kappa) \equiv \mathrm{Span}\{|\Psi^E_i\rangle, \quad i = 1, \ldots, \kappa\} \, .$$

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- Use the Gram matrix $G_{ij} \equiv \langle \Psi^E_i | \Psi^E_j \rangle$ and write

$$\mathbf{Z}_{\text{bulk}} \equiv \mathrm{Tr}_{\mathcal{H}^E_{\mathrm{bulk}}(\kappa)} (K_L K_R) = \left(G^{-1}\right)_{ji} \langle \Psi^E_i | K_L K_R | \Psi^E_j \rangle = \lim_{n \to -1} \left(G^n\right)_{ji} \langle \Psi^E_i | K_L K_R | \Psi^E_j \rangle$$

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Step 1. Compute $\overline{\mathbf{Z}_{\text{bulk}}} = \overline{\mathrm{Tr}_{\mathcal{H}^{E}_{\text{bulk}}(\kappa)}\left(K_L K_R\right)}$ in the leading order in e^{-1/G_N}, verify factorization.

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Step 2. Show that $\overline{\mathbf{Z}^2_{\text{bulk}}} = \overline{\mathbf{Z}_{\text{bulk}}}^2 + O\left(\frac{1}{G_N}\right)$. (see the paper for details)
Factorization in a given microcanonical band trivially extends to factorization of any bulk Hilbert subspace corresponding to a finite range of energies.

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- $2D$ JT gravity case: done by Boruch et. al., 2406.04396 exactly to all orders of G_N.
- **Present work:** generalization to GR in arbitrary $D \geq 3$ in the leading order of e^{-1/G_N}.

Why take the coarse-grained gravity path integral seriously? (at semiclassical level)

1. AdS/CFT-correspondence
We can precisely construct states in CFT which have convenient geometric duals in the leading order of the semiclassical limit. We can then compute correlators in these states such as Z_{bulk} by summing over the saddle point geometries satisfying the appropriate boundary conditions.

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2. Replica wormholes and the unitary Page curve

Recent progress in black hole information problem shows that nontrivial saddle points (replica wormholes) of gravity path integral are crucial for restoring unitarity.

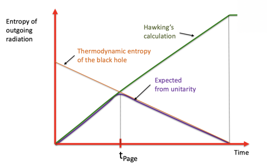

Figure 1: from Almheiri et. al., 2006.06872

Why take the coarse-grained gravity path integral seriously? (at semiclassical level)

3. V. A. did so too

INSTABILITY OF SPACE-TIME DUE TO EUCLIDEAN WORMHOLES

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Abstract

The problem of topology change transitions in quantum gravity is discussed. We argue that the contribution of the Giddings-Strominger wormhole to the Euclidean path integral is pure imaginary. This is checked by two techniques: by the functional integral approach and by the analysis of the Wheeler-De Witt equation. We present also a simple quantum mechanical model which shares many features of the system consisting of parent and baby universes. In this simple model, we show that quantum coherence is completely lost and obtain the equation for the effective density matrix of the "parent universe".

A NEGATIVE MODE ABOUT EUCLIDEAN WORMHOLE

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Abstract

Wormholes – solutions to the euclidean Einstein equations with non-trivial topology – are usually assumed to make real contributions to amplitudes in quantum gravity. However, we find a negative mode among fluctuations about the Giddings-Strominger wormhole solution. Hence, the wormhole contribution to the euclidean functional integral is argued to be purely imaginary rather than real, which suggests the interpretation of the wormhole as describing the instability of a large universe against the emission of baby universes.

Outline of the rest of the talk

1. Review of semiclassical black hole microstates in GR

Balasubramanian et. al., 2212.02447 & 2212.08623

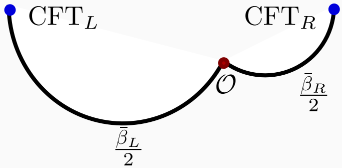
- Microstates as thin shells inside AdS black holes
- Overlap structure
- Gram matrix and the resolvent

2. Trace factorization at the coarse-grained level

- Computing the trace

Semiclassical microstates: boundary view

The boundary state $|\Psi\rangle \in \mathcal{H}_L \otimes \mathcal{H}_R$ can be represented by the Euclidean time contour



Energy basis:

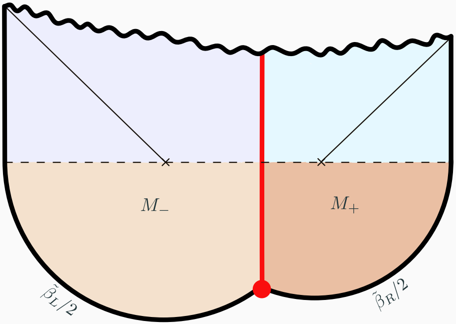
$$H_L|m\rangle_L \otimes |n\rangle_R = E_m|m\rangle_L \otimes |n\rangle_R\,, \qquad H_R|m\rangle_L \otimes |n\rangle_R = E_n|m\rangle_L \otimes |n.\rangle_R$$

The microstate:

$$|\Psi_j\rangle = \frac{1}{\sqrt{Z_1}} \sum_{n,m} e^{-\frac{1}{2}\tilde{\beta}_L E_m - \frac{1}{2}\tilde{\beta}_R E_n} \langle n|\mathcal{O}_j|m\rangle\rangle_L \otimes |n\rangle_R\,, \tag{1}$$

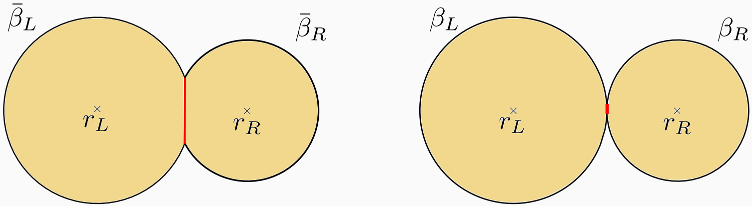
where $Z_1 = \mathrm{Tr}\left[\mathrm{\mathcal{O}_j^\dagger e^{-\tilde{\beta}_L H} \mathcal{O}_j e^{-\tilde{\beta}_R H}}\right]$.

Semiclassical microstates: bulk view



The operator $\mathcal{O}_j$ is dual to a shell of mass determined by $m_j^2 = j^2\mu^2 = \Delta_j(\Delta_j - d)$, where $d = D - 1$.

- Shells are inside both event horizons, they can be arbitrarily heavy
- One can have microstates with multiple shells
- Single-shell states are sufficient to have an overcomplete basis

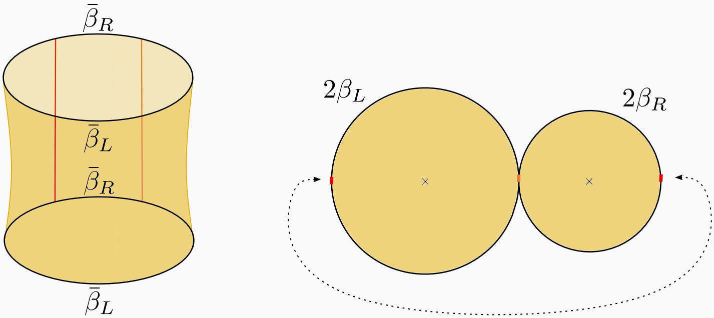


Assumption: large shell mass limit $\mu \to \infty$. This shrinks the shell worldline and pinches off the two Euclidean AdS black hole segments into two disks

$$Z_{\text{on-shell}}[\Psi] = Z(\beta_L)Z(\beta_R)\,, \qquad Z(\beta) = e^{S_{BH}-\beta M}\,,$$

Second moment $\overline{\langle\Psi_i|\Psi_j\rangle\langle\Psi_j|\Psi_i\rangle}$

Key point: there is a connected contribution given by two-boundary wormhole



In the pinching limit, the result is (no sum)

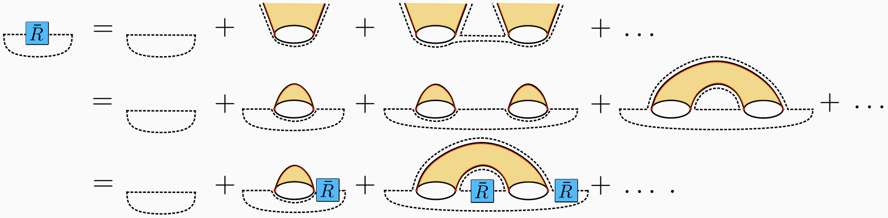
$$\overline{|\langle\Psi_i|\Psi_j\rangle|^2} = \delta_{ij} + \frac{Z(2\beta_L)Z(2\beta_R)}{Z(\beta_L)^2 Z(\beta_R)^2}\,.$$

Resolvent of the microstate Gram matrix

For the Gram matrix $G_{ij} = \langle \Psi_i | \Psi_j \rangle$ we define the (coarse-grained) resolvent

$$\overline{R_{ij}(\lambda)} = \frac{1}{\lambda}\delta_{ij} + \sum_{n=1}^{\infty} \frac{1}{\lambda^{n+1}} \overline{(G^n)_{ij}}$$

The trace of the resolvent $\overline{R} = \sum_{j=1}^{\kappa} \overline{R_{jj}}$ can be represented diagrammatically and resummed as



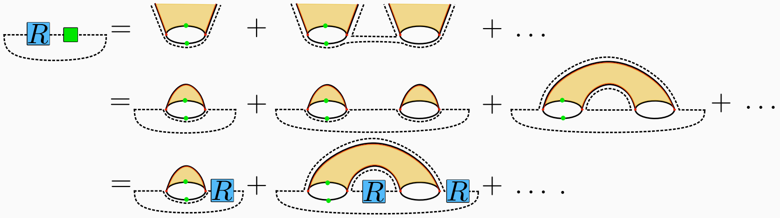
The Schwinger-Dyson equation is easily solved in microcanonical ensemble.

Computing the bulk trace

Coupling back to the bulk trace, we have

$$\overline{\mathbf{Z}_{\text{bulk}}} = \lim_{n\to -1} \overline{(G^n)_{ji}\,\langle \Psi_i^E | K_L K_R | \Psi_j^E \rangle} = \frac{1}{2\pi i}\oint \frac{d\lambda}{\lambda} \overline{R_{ji}(\lambda) \langle \Psi_i^E | K_L K_R | \Psi_j^E \rangle}\,.$$

The r.h.s. can also be represented diagrammatically and resummed:



Result for the bulk trace

- For $\kappa > e^{S_L+S_R}$, the degeneracy of the zero eigenvalue of the Gram matrix G is $\kappa - e^{S_L+S_R}$, and the resolvent solution satisfies $\mathrm{Res}_{\lambda=0} R(\lambda) = \kappa - e^{S_L+S_R}$. The bulk trace gives a factorized result:

$$\overline{\mathrm{Tr}_{\mathcal{H}^E_{\mathrm{bulk}}(\kappa)}(K_L K_R)} = \overline{\mathbf{Z}_{\mathrm{bulk}}} = e^{S_L+S_R-\beta_L E_L-\beta_R E_R}\,.$$

- For $\kappa < e^{S_L+S_R}$, the Gram matrix has no zero eigenvalue and thus $R(0) = R_0$ with R_0 a finite number. The bulk trace does not factorize:

$$\overline{\mathrm{Tr}_{\mathcal{H}^E_{\mathrm{bulk}}(\kappa)}(K_L K_R)} = e^{-\beta_L E_L-\beta_R E_R}\,\frac{R_0 e^{S_L+S_R}}{R_0 - e^{S_L+S_R}}\,.$$

Conclusions

- We have shown that bulk Hilbert space of every theory of quantum gravity in AdS which has GR as its low-energy limit, factorizes in the leading order of semiclassical expansion.
- The crucial ingredient is an overcomplete family of semiclassical states with nonperturbatively small overlaps.

Limitations and possible generalizations:

- Leading order of semiclassical expansion
- Gauge symmetries in the bulk require physical states in every irrep of the bulk gauge group for the factorization argument to hold.
- The argument can be generalized to asymptotically flat spacetimes.

Backup

Dynamics of thin shells in Euclidean AdS

Euclidean AdS black hole metric:

$$ds_\pm^2 = f_\pm(r)d\tau_\pm^2 + \frac{dr^2}{f_\pm(r)} + r^2 d\Omega_{d-1}^2 \, ,$$

where

$$f_\pm(r) = \begin{cases} r^2 + 1 - \frac{16\pi G M_\pm}{(d-1)V_\Omega r^{d-2}}, & \text{for } d > 2, \\ r^2 - 8\pi G M_\pm, & \text{for } d = 2. \end{cases}$$

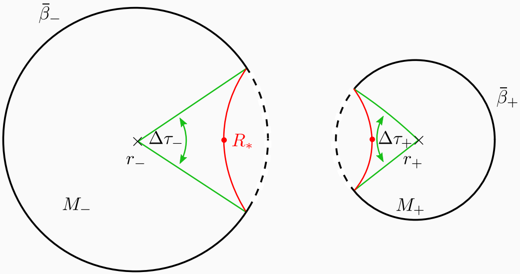
Trajectory of a thin spherical shell parametrized by functions $r = R(T)$, $\tau = \tau_\pm(T)$ is determined by Israel junction conditions:

$$f_\pm \dot{\tau}_\pm = \pm\sqrt{-\dot{R}^2 + f_\pm} \, ,$$
$$\dot{R}^2 + V_{\text{eff}}(R) = 0 \, ,$$

where

$$V_{\text{eff}}(R) = -f_+(R) + \left(\frac{M_+ - M_-}{m} - \frac{4\pi G m}{(d-1)V_\Omega R^{d-2}} \right)^2 \, .$$

Thin shell inside the horizons

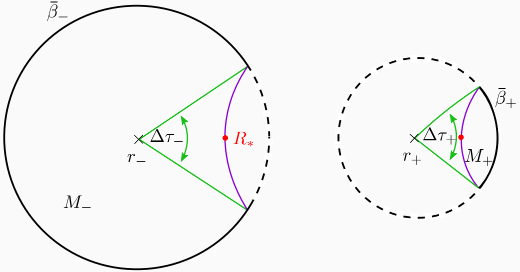


Boundary conditions for the shell inside both horizons: $\tau_+(R_*) = \frac{\beta_+}{2}$, $\tau_-(R_*) = 0$.

The trajectory is given by

$$\begin{aligned}\tau_-(R) &= \int_{R_*}^{R} \frac{d\tilde R}{f_-}\sqrt{\frac{f_- + V_{\text{eff}}}{-V_{\text{eff}}}},\\ \tau_+(R) &= \frac{\beta_+}{2} + \int_{R_*}^{R} \frac{d\tilde R}{f_+}\sqrt{\frac{f_+ + V_{\text{eff}}}{-V_{\text{eff}}}}.\end{aligned}$$

Thin shell outside the r_+ horizon



Boundary conditions for the shell infalling from outside the r_+ horizon:
$\tau_+(R_*) = 0$, $\tau_-(R_*) = 0$.
The trajectory is given by

$$\begin{aligned} \tau_-(R) &= \int_{R_*}^{R} \frac{d\tilde{R}}{f_-}\sqrt{\frac{f_- + V_{\text{eff}}}{-V_{\text{eff}}}}\,, \\ \tau_+(R) &= \int_{R_*}^{R} \frac{d\tilde{R}}{f_+}\sqrt{\frac{f_+ + V_{\text{eff}}}{-V_{\text{eff}}}}\,. \end{aligned}$$

Problem of black hole microstate (over)counting

Black holes have finite entropy:

$$S_{BH} = \frac{A}{4G}\,,$$

In quantum gravity we expect it to be equal to the coarse-grained entropy

$$S = \dim \mathcal{H}$$

- Top-down string models (Strominger, Vafa) and fuzzball paradigm produce quantum gravity microstates which reproduce the Bekenstein-Hawking entropy.
- But can this microstate counting be found in any semiclassical gravity?

Balasubramanian et. al., 2212.02447 & 2212.08623: there is infinite set of microstates in semiclassical gravity which counts the coarse-grained entropy matching with Bekenstein-Hawking.

Microstate counting: main idea

We study the Gram matrix of microstates: $G_{ij} = \langle \Psi_i | \Psi_j \rangle$; $i, j = \overline{1, \Omega}$
The computation of the overlap structure is performed the gravitational path integral rules, which is denoted by the overline. The main steps:

1. Obtain the expression for the moments of overlaps

$$\overline{G_{ij}^n} = \overline{\langle \Psi_i | \Psi_j \rangle^n}$$

   for general n.

2. Solve the resolvent equation for the Gram matrix:

$$\overline{R_{ij}(\lambda)} := \overline{\left(\frac{1}{\lambda \mathbf{1} - G} \right)_{ij}} = \frac{1}{\lambda} \delta_{ij} + \sum_{n=1}^{\infty} \frac{1}{\lambda^{n+1}} \overline{(G^n)_{ij}}\,.$$

3. Find the density of eigenvalues of the Gram matrix in the gravity path integral, which is defined as

$$\overline{D(\lambda)} = \lim_{\epsilon \to 0} \frac{1}{2\pi i} \overline{(R(\lambda - i\epsilon) - R(\lambda + i\epsilon))}\,.$$