# The CBK relation in QCD : the current status of considerations

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# Scientific subjects to be discussed

- Axial-Vector-Vector anomaly as the bridge between Deep Inelastic and Annihilation procecceses ;
- Bridges between DI sum rules and $e^+e^-$-annihilation to hadrons D-function (and R-ratio) PT QCD expressions ;
- Tree branches and Conformal symmetry level related identical (!) cancellations of the perturbative QCD effects ;
- High orders perturbative QCD and Renormalization Group related signals of the conformal symmetry violations ;
- Perturbative QCD (scale-scheme) ambiguities ; Concrete manifestations within $\overline{\text{MS}}$-scheme vs Principle of Maximal Conformality ; Where are the Landau pole effect (?) ;
- Concluding comments or Moondust arguments for perturbative studies of Supersymmetry Gauge Models as the reply to discussions with V.A.Rubakov

# Definitions of basic quantities in the relations

The $e^+e^-$ to hadrons D function in QCD with $a_s = \alpha_s/\pi$

$$D(a_s(Q^2)) = Q^2 \int\limits_0^\infty ds \frac{R^{th}_{e^+e^-}(a_s(s))}{(s+Q^2)^2} \to Q^2 \int\limits_0^\infty ds \frac{R^{exp+th}_{e^+e^-}(s)}{(s+Q^2)^2},$$

$R^{th}_{e^+e^-} = \sigma_{tot}^{e^+e^- \to hadrons}(a_s)/(\sigma_0(e^+e^- \to \mu^+\mu^-) = 4\pi\alpha/(3s)); \alpha\ ; \sigma_0??$

$$\left(\frac{\partial}{\partial \ln \mu^2} + \beta(a_s)\frac{\partial}{\partial a_s}\right) D(a_s) = 0, \frac{\partial a_s}{\partial \ln \mu^2} = \beta(a_s)$$

$$D\left(a_s\right) = \left(\sum_i q_i^2\right) D^{NS}\left(a_s\right) + \left(\sum_i q_i\right)^2 D^{SI}\left(a_s\right)$$

INR and JINR long termed project Chetyrkin,Kataev,Tkachov (79)-Gorishny,Kataev,Larin (87-91)- Baikov, Chetyrkin, Kuhn (2010,2012) + various confirmations of 1979,1991,2010

# Definitions of basic quantities in the relations

$$S_{Bjp}(Q^2) = \int_0^1 dx[g_1^{(lp)}(x, Q^2) - g_1^{(ln)}(x, Q^2)] = \frac{1}{6}|\frac{g_A}{g_V}|C_{Bjp}(a_s(Q^2))$$

$$C_{Bjp}(a_s) = C_{Bjp}^{NS}(a_s) + d_R\left(\sum_f Q_f\right)C_{Bjp}^{SI}(a_s)$$

Another INR and JINR long turmed project : Gorishny,Larin (85)-Larin, Vermaseren (91)- Baikov,Chetykin,Kuhn (2010)

QCD $\beta$-function third JINR and INR long turmed project : Tarasov, Vladimirov, Zharkov (1980) ; Ritbergen,Vermaseren, Larin (1997); Baikov, Chetyrkin , Kühn, (2017) and the number of confirmations

# Definitions and the CBK relation key findings

$$D^{NS}(a_s) = 1 + \sum_{k\geq 1} D_k a_s^k \ , \ C^{NS}_{Bjp}(a_s) = 1 + \sum_{k\geq 1} C_k a_s^k \ , D_1 = -C_1$$

$$\beta(a_s) = -\sum_{k\geq 0} \beta_k a_s^{k+2}$$

Crewther (1972) and Broadhurst, Kataev (1993) CBK relation

$$C^{NS}_{Bjp}(a_s) D^{NS}(a_s) = 1 + ZERO(a_s) + \left(\frac{\beta(a_s)}{a_s}\right) \sum_{k\geq 1} K_n a_s^n =$$

$$= (K, Mikhailov(10-12)) = 1 + \sum_{n\geq 0} \left(\frac{\beta(a_s)}{a_s}\right)^n P_n(a_s)$$

$ZERO(a_s) = P_0(a_s) = 0$ ; Conformal symmetry manifestations in the AVV diagram renormalizations. Related to Adler-Bardeen non-renormalization theorem ; Facorization of $\beta$-function in all orders Crewther (97), Braun,Korchemsky,Mueller (2003) ; Valid in Gauge Invariant DIAGRAMMATIC renormalization schemes

# Theory aspects of the CBK relation


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# Recent CBK related studies


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# Reminding Note

Conformal Symmetry is the symmetry under the following transformations of coordinates

- Scale transformation or dilatation $x^{'\mu} = \rho x^{\mu}$ with 1 parameter $\rho > 0$
- Special conformal transformation $x^{'\mu} = \frac{x^{\mu}+\beta^{\mu}x^2}{1+2\beta x+\beta^2 x^2}$ with 4 parameters $\beta^{\mu}$
- Translations $x^{'\mu} = x^{\mu} + \alpha^{\mu}$ with 4 parameters $\alpha^{\mu}$
- Homogeneous Lorentz transformations $x^{'\mu} = \Lambda^{\mu}_{\nu} x^{\nu}$ that also contain 4 parameters

The CS symmetry within PT is violated by the procedure of renormalizations in the renormalizable theories and leads to the appearence of Conformal anomaly- The term $(\beta(a)/a)$ which is appearing before Trace of Energy Momentum Tensor $(\beta(a) = -\sum_{i\geq 0}\beta_i a^{i+2}$ is the perturbatrive RG $\beta$-function

# The conctete $e^+e^-$ annihilation D-function case

We will follow AK,Molokoedov PRD 108, 096027 (2023) $\overline{\text{MS}}$-expressions

$$D_{\overline{\text{MS}}}(a_s) = 3\sum_f Q_f^2\big(1 + a_s + (1.9857 - 0.1153)a_s^2$$
$$+(18.243 - 4.216n_f + 0.086n_f^2 - 0.413\delta_f)a_s^3$$
$$+(135.792 - 34.440n_f + 1.875n_f^2 - 0.010n_f^3 - (5.942 - 0.192n_f)\delta_f)a_s^4\big)$$

$$D_{PMC/BLM}(a_s) = 3\sum_f Q_f^2\big(1 + a_* + \frac{1}{12}a_*^2 + (-23.223 - 0.413\delta_f)a_*^3$$
$$+(81.157 - 0.0080n_f - 2.780\delta_f)a_*^4\big)$$

Note that $a_* = a_s(Q^2/\Lambda^{BLM}(n_f))$

# PMC/BLM vs massless $\overline{\mathrm{MS}}$: AK,Molokoedov PRD(23)

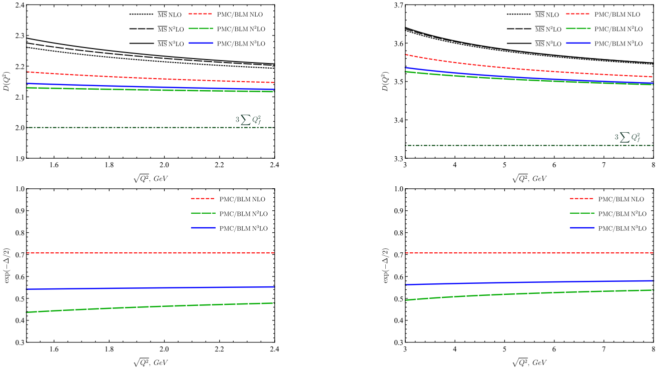


Figure: (1a) Adler function $D(Q^2)$ on $\sqrt{Q^2}$ at $n_f = 3, 4$ in the massless limit. (1b) PMC/BLM Factor $\exp(-\Delta/2)$ on $\sqrt{Q^2}$ . Experimental related data higher (!) $\overline{MS}$ Eidelman, Jegerlehner, K, Veretin (98); Davier et al (23). $\overline{\mathrm{MS}}$ more preferable from the point of view of the experimentally related EJKV data **which is in low energy region is significantly HIGHER** .

# Questions and ambiguites

Where is the IR Landau pole in the PMC/BLM ? Or how its removed in $a_* = a_s(Q^2/\Lambda^{BLM}(n_f))$ ?
Answer : by the proportional to powers of $\beta_0 a_s$ shifts of scales $\Lambda \to \Lambda^{PMC/BLM}$

$$a_* = a_s(Q^2/\Lambda^{BLM}(n_f)) \ \ where \ \ \Lambda^{BLM}(n_f) = \Lambda_{NLO} exp[-\frac{1}{12}\Delta_0]$$

$$\Lambda^{PMC/BLM}(n_f) = \Lambda_{NLO} exp[-\frac{1}{12}\Delta_0 + \beta_0\Delta_1 a_s^{BLM}(Q^2/\Lambda^2_{BLM})]$$

$$\Lambda^{PMC/BLM}(n_f) = \Lambda_{NNLO} exp[-\frac{1}{12}\Delta_0 +$$

$$+\beta_0\Delta_1 a_s^{PMC/BLM}(Q^2/\Lambda^2_{PMC/BLM}) + \beta_0^2\Delta_2(a_s^{BLM}(Q^2/\Lambda^2_{PMC/BLM}))^2]$$

Finally, there is **not yet fixed ambiguity due to the fact that** in the related to $a_s$ and $a_*$ defined in PMC/BLM procedure $\beta_k^{PMC/BLM} \neq \beta_k$ coefficients for the QCD $\beta$-function at $k \geq 2$. To our nowledge this ansertainty WAS NOT YET taken intom account.

# Different representations for the $D^{NS}(a_s)$ in QCD

It is possible to reqrite RG expressions the $e^+e^- \to hadrons$ D-function as

$$D^{NS}(a_s) = 1 + \sum_{k\geq 1} D_k a_s^k = \gamma_{ph}(a_s) - \beta(a_s)\frac{\partial}{\partial a_s}\Pi(a_s)$$

$$= \gamma_{ph}(a_{**})$$

Where $\gamma_{ph}(a_s) = \sum_{k\geq 0}\gamma_k a_s^k$ and $\Pi(a_s) = \sum_{k\geq 1}\Pi_k a_s^k$. Note that starting from three-loop $\gamma_{k\geq 2} = \gamma_{k\geq 2}(n_f)$ DO DEPEND from $n_f$. This represntation was used for PMC/BLM-type analysis S. J. Brodsky, M. Mojaza, X. G. Wu, Phys. Rev. D **89** 014027 (2014).

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**QCD variant of N=1 SUSY QCD Adler function**

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# Conclusions

There are still a lot of interesting problems to study in perturbative QCD . Amonst them are

- Manifesttaion of SUSY QCD structures of the PT series in the real world
  **Due to terminology of friendly discusions one at the Seminars at ITEP with V.Rubakov and A.Zakharov these are SUSY tarces in the Moondust of QCD**
- It is possible that these Moondust QCD traces may be better understood by approaching closer to the considerd at S. Mooij and M. Shaposhnikov, Nucl. Phys. B **990**, 116172 (2023) Callen renormalization scheme
  *But this may be another story*