On the $K^+ \to \pi^0\pi^0\pi^0 e^+\nu$ decay

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PDG/OKA/ChPT results

- OKA recent result: $Br\left(K^+ \to \pi^0\pi^0\pi^0 e^+\nu\right) < 5.4 \times 10^{-8}$ at 90% confidence level. (arXiv:2409.08817)
- PDG result: $Br\left(K^+ \to \pi^0\pi^0\pi^0 e^+\nu\right) < 3.5 \times 10^{-6}$ at 90% confidence level.
- Chiral perturbation theory (Blaser): $Br\left(K^+ \to \pi^0\pi^0\pi^0 e^+\nu\right) = 2.5 \times 10^{-12}$.
- Can the OKA upper bound become closer to the theoretical prediction?

Four-particle vs five-particle phase volume

- It was shown by Blaser (Physics Letters B 345 (1995) 287-290) that

$$\left.\frac{\Gamma\left(K^+\to\pi^0\pi^0e^+\nu_e\right)}{\Gamma(K^+\to\pi^0\pi^0\pi^0e^+\nu_e)}\right|_{tree}\approx 3.4\times 10^6.$$

- At the same time as it will be shown

$$m_K^2\frac{V_4/2!}{V_5/3!}\approx 2.1\times 10^6,$$

  where V_n is a n-particle phase volume.
- Consequently the main difference comes from the phase volumes ratio.

Pionium $A_{2\pi}$

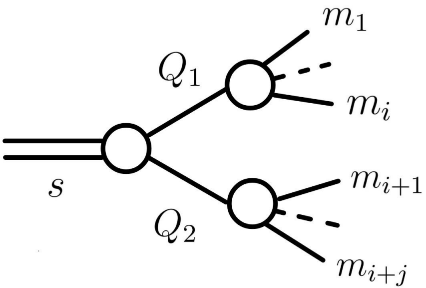
- $A_{2\pi}$ is a $\pi^+\pi^-$ atom bound by e/m interaction.
- Decay length $c\tau \sim 10^{-4}$cm is much larger than nuclear interaction scale.
- Main decay channel is $A_{2\pi} \to \pi^0\pi^0$.
- If the process $K^+ \to A_{2\pi}\pi^0 e^+\nu_e$ occurs then V_5 is replaced by V_4 and it enhances the width.

Recurrent formula

- For the V_n the following relation can be proved:

$$dV_n(s; m_1^2, m_2^2, ..., m_n^2) = \int \frac{dQ_1^2}{2\pi}\frac{dQ_2^2}{2\pi} dV_2(s; Q_1^2, Q_2^2) \times dV_i(Q_1^2; m_1^2, ..., m_i^2) \times dV_j(Q_2^2; m_{i+1}^2, ..., m_{i+j}^2).$$

- It is convenient to use the following graphs:



V_4 and V_5

- For $K^+ \to \pi^0\pi^0 e^+\nu_e$ the recurrent formula leads to

$$V_4(s = m_K^2; 0, 0, m_{\pi^0}^2, m_{\pi^0}^2) = \int \frac{dQ_1^2}{2\pi}\frac{dQ_2^2}{2\pi}\frac{\sqrt{\left[s - \left(\sqrt{Q_1^2} - \sqrt{Q_2^2}\right)^2\right]\left[s - \left(\sqrt{Q_1^2} + \sqrt{Q_2^2}\right)^2\right]}}{8\pi s} \times \frac{1}{8\pi} \times \frac{\sqrt{Q_2^2(Q_2^2 - 4m_\pi^2)}}{8\pi Q_2^2}.$$

- For the $K^+ \to \pi^0\pi^0\pi^0 e^+\nu_e$ in the same way we obtain

$$V_5(m_K^2; 0, 0, m_{\pi^0}^2, m_{\pi^0}^2, m_{\pi^0}^2) = \int \frac{dQ_1^2}{2\pi}\frac{dQ_2^2}{2\pi}\frac{\sqrt{\left[s - \left(\sqrt{Q_1^2} - \sqrt{Q_2^2}\right)^2\right]\left[s - \left(\sqrt{Q_1^2} + \sqrt{Q_2^2}\right)^2\right]}}{8\pi s} \times \frac{1}{8\pi} \times \frac{\left(\sqrt{Q_2^2} - 3m_{\pi^0}\right)^2}{2^6 3\sqrt{3}\pi^2}.$$

- Integration limits:

$$0 < Q_1^2 < (m_K - nm_{\pi^0})^2,$$
$$(nm_{\pi^0})^2 < Q_2^2 < \left(m_K - \sqrt{Q_1^2}\right)^2,$$

where $n = 4, 5$ is the number of pions in the final state.

V_4 and V_5

Approximations used:

1. $m_e = 0 \rightarrow V_2 = \frac{1}{8\pi}$.
2. Pions in $Ke5$ decay are non-relativistic $\rightarrow V_3 = \frac{\left(\sqrt{s} - 3m_{\pi^0}\right)^2}{2^6 3\sqrt{3}\pi^2}$.

As a result of integration:

- $V_4 = 7.1 \times 10^{-3} \frac{m_K^4}{2^{11}\pi^5}$.
- $V_5 = 1.37 \times 10^{-6} \frac{m_K^6}{2^{14} 3\sqrt{3}\pi^6}$.
- Taking into account identity of neutral pions we obtain

$$m_K^2 \frac{V_4/2!}{V_5/3!} = 2.1 \times 10^6.$$

$A_{2\pi}\pi^0 e\nu_e$ in the final state

- In this case we have again the V_4 with substitution $m_{\pi^0} \to 2m_{\pi^+}$:

$$V_A = \int \frac{dQ_1^2}{2\pi}\frac{dQ_2^2}{2\pi}\frac{\sqrt{\left[s-\left(\sqrt{Q_1^2}-\sqrt{Q_2^2}\right)^2\right]\left[s-\left(\sqrt{Q_1^2}+\sqrt{Q_2^2}\right)^2\right]}}{8\pi s}\times\frac{1}{8\pi}\times$$
$$\times\frac{\sqrt{\left[Q_2^2-\left(\sqrt{4m_{\pi^+}^2}-\sqrt{m_{\pi^0}^2}\right)^2\right]\left[Q_2^2-\left(\sqrt{4m_{\pi^+}^2}+\sqrt{m_{\pi^0}^2}\right)^2\right]}}{8\pi Q_2^2},$$

where the integration limits are

$$0 < Q_1^2 < (m_K - 2m_{\pi^+} - m_{\pi^0})^2,$$
$$(2m_{\pi^+} + m_{\pi^0})^2 < Q_2^2 < \left(m_K - \sqrt{Q_1^2}\right)^2,$$

- After performing integration $V_A = 0.83 \times 10^{-4}\frac{m_K^4}{2^{11}\pi^5}$.

Ratio estimation and $\psi(0)$

- The increase of the width is
$$\frac{m_K^2 V_A}{V_5/3!} \approx 5 \times 10^4.$$
- But the pionium is a bound system $\rightarrow$ we need to modify the amplitude as well:
$$< A_{2\pi}\pi^0 e^+ \nu_e | K > \sim \int d\vec{p}\phi(\vec{p}) < \pi^+\pi^-\pi^0 e^+\nu_e | K > \Big|_{|\vec{p}|\rightarrow 0} \rightarrow \psi(0) < \pi^+\pi^-\pi^0 e^+ \nu_e | K > .$$
- As a result
$$\frac{\Gamma(K^+ \rightarrow A_{2\pi}\pi^0 e^+ \nu_e)}{\Gamma(K^+ \rightarrow \pi^0\pi^0\pi^0 e^+\nu_e)} \sim |\psi(0)|^2 \frac{m_K^2 V_A}{V_5/3!} \sim 7.5 \times 10^{-4},$$
where $\psi(0) = 1/\sqrt{\pi a_B^3} \sim \alpha^{3/2}/(2\sqrt{2\pi})$.
- We see the suppression of the decay via pionium rather than enhancement.

Conclusion

- The impact of pionium into the $Ke5$ decay was investigated.
- The recurrent relations for V_4 and V_5 were used.
- It was shown that in case of $Ke4$ and $Ke5$ the width ratio is determined by phase spaces ratio.
- The pionium width was estimated and the phase space enhancement was shown.
- It was shown that there is additional suppression due to the e/m interaction inside pionium.
- If there was strongly coupled bound state the additional suppression would vanish.