Matrix QM

and

Index theorems

Dmitri Bykov

with V. Krivorol
& A. Kuzovchikov

Steklov Mathematical Institute
ITMP (Moscow State University)

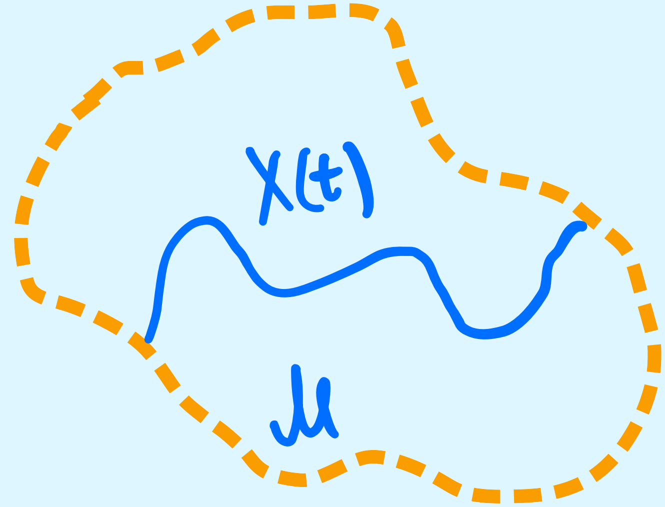
Rubakov-70 RAS session, 18 Feb 2025

Russian Science Foundation

1D sigma models

$$(\mathcal{M} \mid G, A)$$

Manifold Metric Gauge field



$$L = \int dt \, \frac{1}{2} G_{IJ}(x) \, \dot{x}^I \dot{x}^J - \int dt \, A_I(x) \, \dot{x}^I + \text{SUSY}$$

Classical: (Magnetic) geodesics

Quantum: (Generalized) Laplacians Δ

Bochner, Dolbeault, de Rham

Target space

$$\mathcal{M} = \text{Homogeneous} \quad \frac{SU(n)}{H}$$

symplectic
complex
Kähler

(Co)adjoint orbit $\quad g\Lambda g^{-1} \quad \Lambda \in su_n, \; g \in SU(n)$

$$\Lambda = \begin{pmatrix} \boxed{\begin{matrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_1 \end{matrix}}_{\,n_1} & & & \\ & \boxed{\begin{matrix} \lambda_2 & & \\ & \ddots & \\ & & \lambda_2 \end{matrix}}_{\,n_2} & & \\ & & \ddots & \\ & & & \boxed{\begin{matrix} \lambda_m & & \\ & \ddots & \\ & & \lambda_m \end{matrix}}_{\,n_m} \end{pmatrix}$$

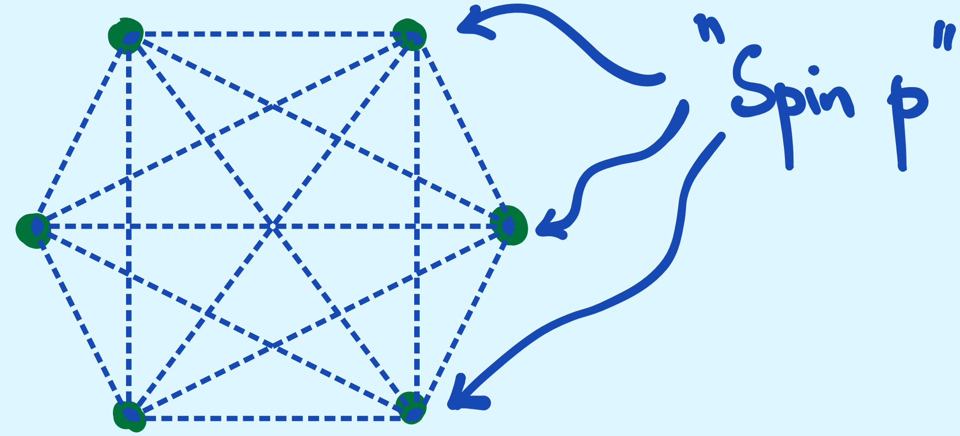
$$\boxed{\frac{SU(n)}{S(U(n_1) \times \cdots \times U(n_m))}}$$

Flag manifold

SUSY Spin chain

$$\Big\downarrow H_p$$

Exact truncation of Δ



* Oscillator variables | Nicolai '1976-77

* Witten index of spin chain | Witten '1982, Alvarez-Gaume '1983

$\rightsquigarrow$ index theorems on flags

$\mathbb{CP}^1 \simeq S^2$

Spectrum: $E_\ell = \ell(\ell+1)$, $\ell = 0, 1, 2, \ldots$

Y_ℓ^m spherical harmonics

$$V_{2\ell} := \underbrace{\square\square\square\square\square\square}_{2\ell} \qquad su_2 \text{ representation}$$

Truncate to first '$p+1$' harmonics $\ell = 0, 1, \ldots, p$

$$\boxed{\bigoplus_{\ell=0}^{p} V_{2\ell} = V_p \otimes V_p}$$

$\Rightarrow$ 'Spin chain' 

Schwinger-Wigner osillators

$q =$ monopole charge

'Spin chain'

V_p $\qquad$ V_{p+q}

Tamm '1931

Wu-Yang '1976

$$\text{Hamiltonian} = \text{Casimir} = \sum_{\alpha=1}^{3} S_{1\alpha} S_{2\alpha} \equiv (S_1, S_2)$$

$$S_{A\alpha} = a^{+}_{Ai} (\sigma_\alpha)_{ij}\, a_{Aj} \qquad [a_{Ai}, a^{+}_{Bj}] = \delta_{AB}\, \delta_{ij}$$

$$a_1^{+} \circ a_1 = p\,, \qquad a_2^{+} \circ a_2 = p + q$$

$$\boxed{H = (a_2^{+} \circ a_1)(a_1^{+} \circ a_2)}$$

1D sigma models with $\mathcal{N}=2$ SUSY

- 2a model / de Rham

'Hull 1999
'Ivanov, Smilga 2012

X^A = real superfields

$$\mathcal{L} = \int d^2\theta \left[G_{AB}\, DX^A \bar{D} X^B - W(X) \right]$$

$$Q \leftrightarrow e^{-W} d\, e^{W}$$

- 2b model / Dolbeault

Z^A = complex chiral superfields

$$\mathcal{L} = \int d^2\theta \left[G_{AB}\, DZ^A \bar{D} \bar{Z}^B - W(Z,\bar{Z}) \right]$$

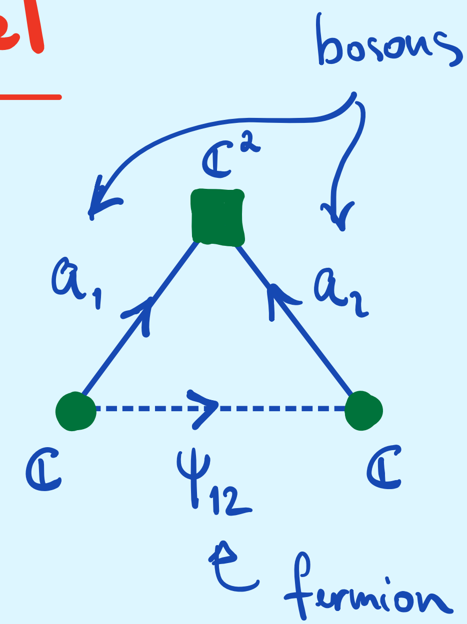
$$Q \leftrightarrow e^{-W} \partial\, e^{W}$$

SUSY extension $\mathcal{N}=2$ / D-model

$$H_{susy} = \{Q, Q^+\}$$

$$Q = \alpha_{12}\, \psi_{12}\, a_1^+ \circ a_2$$

α_{12} ← real parameter

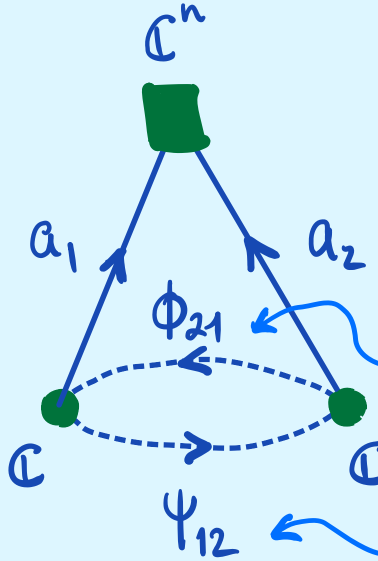


Constraints:
$$C_1 = a_1^+ \circ a_1 + \psi_{12}^+ \psi_{12} - p = 0$$
$$C_2 = a_2^+ \circ a_2 - \psi_{12}^+ \psi_{12} - (p+q) = 0$$

$$\boxed{[C_1, Q] = [C_2, Q] = 0}$$

H_{susy} = Truncation of Dolbeault Δ on $\mathbb{CP}^1$

Kähler-de Rham $\mathcal{N}=4$ / K-model



$$Q_1 = \alpha_{12}\, \psi_{12}\, a_1^\dagger \circ a_2$$

$$Q_2 = \alpha_{12}\, \phi_{21}^\dagger\, a_1^\dagger \circ a_2$$

SU(2) doublet

twice as many fermions: ψ_{12}, ϕ_{21}

No monopoles!

Constraints: $$C_1 = a_1^\dagger \circ a_1 + \psi_{12}^\dagger \psi_{12} - \phi_{21}^\dagger \phi_{21} - p = 0$$

$$C_2 = a_2^\dagger \circ a_2 - \psi_{12}^\dagger \psi_{12} + \phi_{21}^\dagger \phi_{21} - p = 0$$

$$\{Q_A, Q_B\} = 0\,, \quad \{Q_A, Q_B^\dagger\} = \delta_{AB} H$$

Truncation of de Rham Δ on $\mathbb{CP}^1$

D-model in N=2 superspace

Superderivatives $D, \bar{D} \mapsto D^2 = \bar{D}^2 = 0$

Superfields $A_{1i} = a_{1i} + \dots,\ A_{2i} = a_{2i} + \dots$

Coupling

$$\bar{D}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \underbrace{\begin{pmatrix} \Lambda_1 & x\Psi_{12} \\ 0 & \Lambda_2 \end{pmatrix}}_{:= \mathcal{B}} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

"Superconnection"

Flatness: $\bar{D}\mathcal{B} - \mathcal{B}\mathcal{B} = 0$

Ivanov
Krivonos '1997
Toppan

$$L = \int d^2\theta \left[\bar{A}_1 \cdot A_1 + \bar{A}_2 \cdot A_2 + \bar{\Psi}_{12}\Psi_{12} \right] + \text{FI terms}$$

"Free" Lagrangian

$$+\left(p \int d\theta\, \Lambda_1 + (p+q) \int d\theta\, \Lambda_2 + \text{c.c.} \right)$$

K-model in $\mathcal{N}=2$ superspace

Same fields A_{1i}, A_{2i}

Coupling

$$\bar{D}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} \Lambda_1 & \varkappa\Psi_{12} \\ \varkappa\Phi_{21} & \Lambda_2 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

Only $\Lambda_1+\Lambda_2$ is chiral: $\bar{D}(\Lambda_1+\Lambda_2)=0$.

$$L = \int d^2\theta \left[\bar{A}_1\circ A_1 + \bar{A}_2\circ A_2 + \bar{\Psi}_{12}\Psi_{12} + \bar{\Phi}_{21}\Phi_{21} \right] + $$

$$+\, \rho\left(\int d\theta\, (\Lambda_1+\Lambda_2) + \text{c.c.} \right)$$

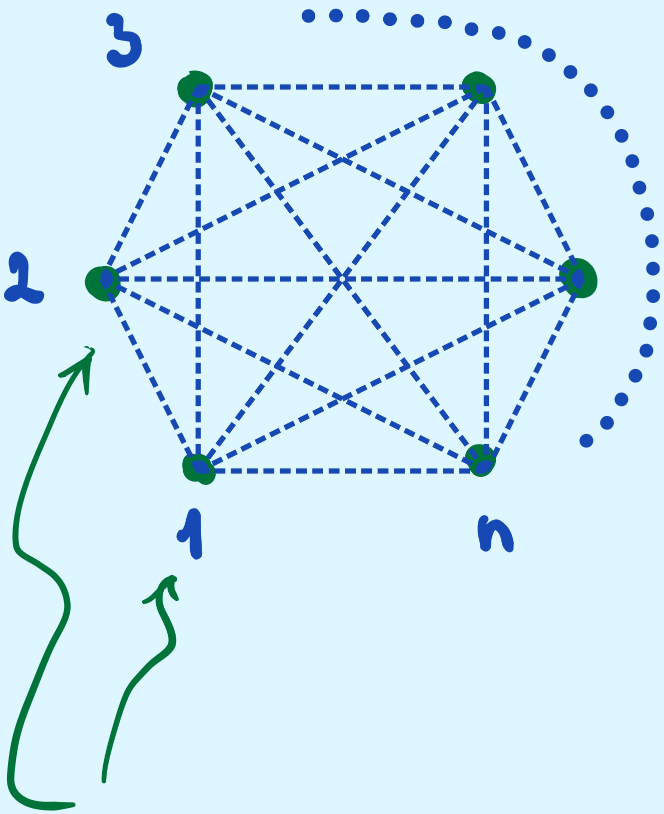
Single FI term

Flags: the spin chain

DB '2024
Kuzovchikov

$$\mathcal{F}_n := \frac{SU(n)}{S(U(1)^n)}$$

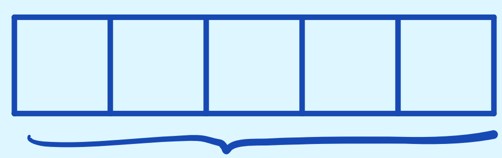
$(n=2: \mathbb{CP}^1)$



$$H = \sum_{A<B} d_{AB}^2 \, (S_A, S_B)$$

$\frac{n(n-1)}{2}$ parameters

Representations $V_{p_1}, V_{p_2} \dots$



$p_i = p + q_i$ magnetic charges

Metric

$$ds^2 = \sum_{A<B} \frac{1}{d_{AB}^2} \left| \bar{u}_A \circ du_B \right|^2$$

$\bar{u}_A \circ u_B = \delta_{AB}$

Supercharges

cubic terms needed for $Q^2=0$

D-model:

$$Q = \sum_{A<B} \alpha_{AB}\, \psi_{AB}\, a_A^{\dagger} \circ a_B \;-\; \sum_{A<B<C} \frac{\alpha_{AB}\,\alpha_{BC}}{\alpha_{AC}}\, \psi_{AB}\, \psi_{BC}\, \psi_{AC}^{\dagger}$$

K-model:

$$\xi_{AB} := \begin{pmatrix} \psi_{AB} \\ \phi_{BA}^{\dagger} \end{pmatrix}$$

SU(2) doublets

$$Q = \sum_{A<B} \alpha_{AB}\, \xi_{AB}\, a_A^{\dagger} \circ a_B + \sum_{A<B<C} \left(\frac{\alpha_{AB}\,\alpha_{AC}}{\alpha_{BC}}\, \xi_{AB} \left(\xi_{AC}^{\dagger}\, \xi_{BC} \right) - \frac{\alpha_{AC}\,\alpha_{BC}}{\alpha_{AB}}\, \xi_{BC} \left(\xi_{AC}^{\dagger}\, \xi_{AB} \right) \right)$$

SUSY algebra $\Rightarrow$ Kähler constraint $\quad \dfrac{1}{\alpha_{AC}^{2}} = \dfrac{1}{\alpha_{AB}^{2}} + \dfrac{1}{\alpha_{BC}^{2}} \quad A<B<C$

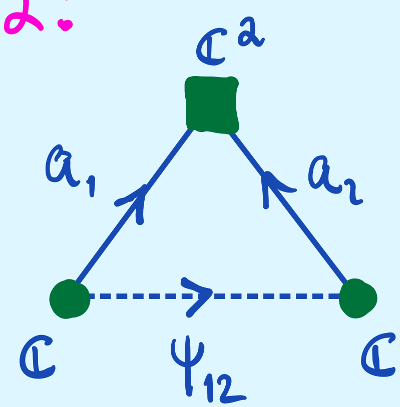
The Witten index / D-model

$$W = \mathrm{STr}\left(g\, e^{-\beta H}\right), \quad g \in SU(n)$$

Independent of β : $\beta \to \infty$ $\quad W = \mathrm{STr}(g)\big|_{H=0}$

$\beta \to 0$ $\quad W = \mathrm{STr}(g)\big|_{\text{constrained Fock space}}$

$n=2$:



$$C_1 = a_1^\dagger \circ a_1 + \psi_{12}^\dagger \psi_{12} - p = 0$$

$$C_2 = a_2^\dagger \circ a_2 - \psi_{12}^\dagger \psi_{12} - (p+q) = 0$$

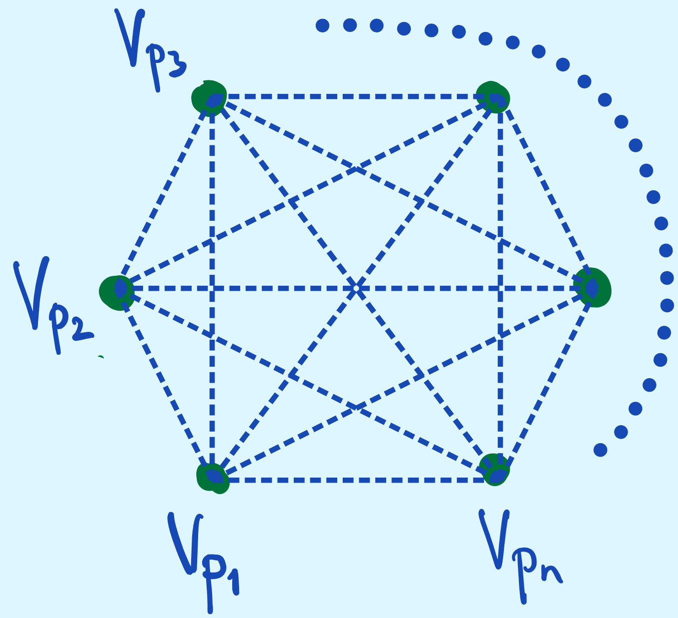
Fermion number 0: $V_p \otimes V_{p+q}$ $\quad$ Fermion number 1: $V_{p-1} \otimes V_{p+q+1}$

$$\boxed{W = \chi_p \chi_{p+q} - \chi_{p-1}\chi_{p+q+1} = \chi_q}$$

Independent of p (of the truncation)

$q+1$ zero energy states

Index: general case



$$p_A = p + q_A$$

$$W = \mathrm{STr}(g)\Big|_{\text{constrained Fock space}} = \,?$$

1) Oscillator partition function

constraints

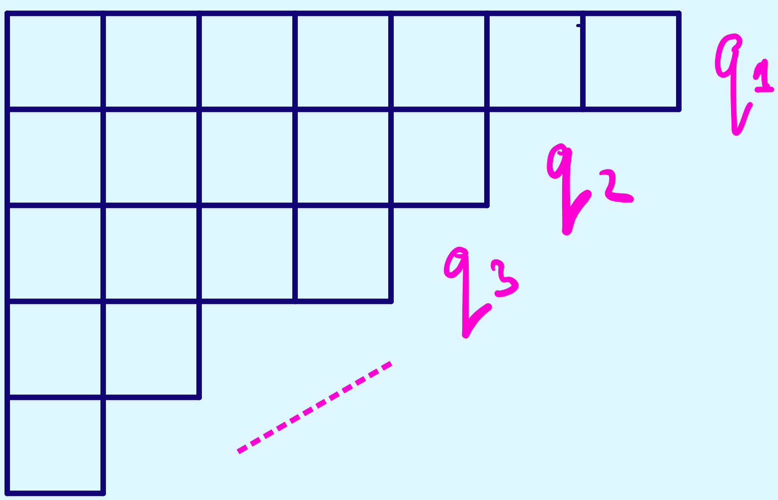
$u(1)^n \subset u(n)$ generators

$$Z(t|s) = \mathrm{STr}\Big(\underbrace{\prod_{i=1}^{n} t_i^{J_i}}_{=g} \prod_{j=1}^{n} s_j^{c_j}\Big) =$$

$$= \prod_{i,j=1}^{n} \frac{1}{1-t_i s_j} \times \prod_{k<l}\left(1-\frac{s_k}{s_l}\right)\frac{1}{s_1^{p_1} s_2^{p_2} s_3^{p_3}}$$

2) Set $C_j = 0$ by residues

$$W = \oint \frac{ds_1}{2\pi i s_1} \cdots \oint \frac{ds_n}{2\pi i s_n} \, Z(t|s) = \text{Weyl formula for } \chi_{q_1,\dots,q_n}$$



Borel-Bott-Weil theorem

K-model: W = Euler characteristic $= n!$

Conclusion & Outlook

| Yamaguchi '1979
| Kuwabara '1988

* Computing spectra of (magnetic) $\nabla \Delta$ on flags

$\equiv$ Diagonalizing spin chains

(infinite spin limit $p \mapsto \infty$)

* SUSY extension: nonlinear chiral multiplets

| Ivanov, Krivonos, Toppan '1997

* Index theorems via oscillator partition function

* Is this an integrable problem?

  Explicit solution?

  | DB, Kuzovchikov '2024

* Generalization to ∞-dim. groups

  (loop groups etc.)

* Relation to 2D sigma models

  (Gross-Neveu models, ...)

  | DB '2020+

THANK
YOU!