РОССИЙСКАЯ АКАДЕМИЯ НАУК

Section of Nuclear Physics of the Department of Physical Sciences of the Russian Academy of Sciences dedicated to the 70th anniversary of the birth of Valery Anatolievich Rubakov

Chiral 3-loop quantum correction to $\mathcal{N}$=1 Wess-Zumino model

work of R. Iakhibbaev, D. Kazakov, I. Buchbinder, D. Tolkachev

BLTP JINR

RAS
2025

Setup: massless $\mathcal{N}$=1 WZ model

Buchbinder, Petrov et al '94
Buchbinder, Kuzenko'98

- Action:

Kahler potential

Chiral potential
with complex coupling constant

$$\mathcal{S}[\Phi,\bar{\Phi}] = \int d^8z\ \Phi\bar{\Phi} + \frac{\lambda}{3!}\int d^6z\ \Phi^3 + \frac{\bar{\lambda}}{3!}\int d^6\bar{z}\ \bar{\Phi}^3$$

contains quartic and Yukawa interactions

Super-measure

$$d^8z = d^4x d^2\theta d^2\bar{\theta}$$
$$d^6z = d^4x d^2\theta$$
$$d^6\bar{z} = d^4x d^2\bar{\theta}$$

Quantum corrections can supplement:

- Kahlerian part (from the first loop)
- Auxiliary field part
- Chiral part**

$$\Gamma[\Phi,\bar{\Phi}] = \sum_{L=1}^{\infty}\hbar^L\Gamma^{(L)}[\Phi,\bar{\Phi}]$$

$$\mathbf{K} = \sum_L \hbar^L \mathbf{K}^{(L)}(\Phi\bar{\Phi}) \qquad \mathbf{A} = \sum_L \hbar^L \mathbf{A}^{(L)}(D\Phi,\bar{D}\bar{\Phi})$$

Non-renormalization theorem:
all the loop corrections to the effective action in such a theories are expressed by integrals over the whole superspace, but not over its chiral subspace.

$$\mathbf{W} = \sum_L \hbar^L \mathbf{W}^{(L)}(\Phi)$$

Buchbinder, Petrov et al '94
Buchbinder, Kuzenko'98

Setup: massless $\mathcal{N}$=1 WZ model

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Super-measure

$$d^8z = d^4x d^2\theta d^2\bar{\theta}$$
$$d^6z = d^4x d^2\theta$$
$$d^6\bar{z} = d^4x d^2\bar{\theta}$$

Anyway, there is a loophole:

$$D^2\bar{D}^2D^2 = 16D^2\partial^2$$

$$\int d^8z\ u(\Phi)\left(-\frac{D^2}{4\partial^2}\right)v(\Phi) = \int d^6z\ u(\Phi)v(\Phi) \quad (*)$$

Possible only for massless diagrams

Therefore, one can obtain finite (three-point functions do not diverge) corrections to the chiral effective potential

$$\mathbf{W} = \sum_L \hbar^L \mathbf{W}^{(L)}(\Phi)$$

this finite terms are non-local in coordinate space

Buchbinder, Petrov et al '94

Effective $\mathcal{N}$=1 superpotential

Effective 1PI action after Legendre transformation of generating functional is

$$\exp\left(\frac{i}{\hbar}\Gamma[\Phi,\bar{\Phi}]\right) = \int \mathcal{D}\phi\mathcal{D}\bar{\phi}\ \exp\frac{i}{\hbar}\mathcal{S}[\Phi+\sqrt{\hbar}\phi,\bar{\Phi}+\sqrt{\hbar}\bar{\phi}]-\left(\sqrt{\hbar}\int d^6z\ \phi(z)\frac{\delta\bar{\Gamma}}{\delta\Phi(z)}+h.c.\right)$$

$$\mathcal{S}^{(2)}[\phi,\bar{\phi}] = \int d^8z\ \phi\bar{\phi} + \left(\frac{1}{2}\int d^6z\ \lambda\Phi\phi^2 + h.c.\right) \qquad \Gamma[\Phi,\bar{\Phi}] = \mathcal{S}[\Phi,\bar{\Phi}] + \bar{\Gamma}[\Phi,\bar{\Phi}] \qquad \Gamma[\Phi,\bar{\Phi}] = \sum_{L=1}^{\infty}\hbar^L\Gamma^{(L)}[\Phi,\bar{\Phi}]$$

Superpropagator:

Not suitable: matrix superpropagator contains effective mass

$$\begin{pmatrix} -\lambda\Phi & \frac{1}{4}\bar{D}^2 \\ \frac{1}{4}D^2 & -\lambda\bar{\Phi}\end{pmatrix}\begin{pmatrix} G_{++} & G_{+-} \\ G_{-+} & G_{--}\end{pmatrix} = -\begin{pmatrix}\delta_+ & 0 \\ 0 & \delta_-\end{pmatrix}$$

$$\Gamma[\Phi,\bar{\Phi}] = \int d^8z\ \left(\mathbf{K}(\Phi,\ \bar{\Phi}) + \mathbf{A}(D\Phi,\bar{D}\bar{\Phi})\right) + \left(\int d^6z\ \mathbf{W}(\Phi) + c.c.\right)$$

Kahler potential — Auxiliary field potential — Chiral potential

condition imposed on effective action

$$\partial_a\Phi = \partial_a\bar{\Phi} = 0$$

Buchbinder, Petrov et al '94

Effective $\mathcal{N}$=1 chiral superpotential

- Improved Feynman super-rules:

indifferent to chirality

$$\mathcal{G}_{ab} = \frac{1}{16}\mathbf{D}\bigoplus G_{ab}$$

improved Green's function

$$(\partial^2 - \frac{1}{4}\bar{\lambda}\bar{\Phi}D^2 - \frac{1}{4}\lambda\Phi\bar{D}^2)\mathcal{G}(z_1, z_2) = \delta^8(z_1 - z_2)$$

- Real superfield propagator:

$$\mathcal{G}(z_1, z_2) = \frac{1}{16}\begin{pmatrix} 0 & -\frac{\bar{D}_1^2 D_2^2}{\partial_1^2}\delta^8(z_1 - z_2) \\ -\frac{D_1^2 \bar{D}_2^2}{\partial_1^2}\delta^8(z_1 - z_2) & -\frac{D_1^2 D_2^2}{\partial_1^2}\left(\lambda\Phi(z_1)\frac{\bar{D}_1^2}{4\partial_1^2}\delta^8(z_1 - z_2)\right) \end{pmatrix}$$

Two-loop diagram example:

Covariant superderivative turned to integration point

$$\Gamma^{(2)}(\Phi) = \frac{\bar{\lambda}^2}{3! \times 2}\int d^6\bar{z}_1 d^6\bar{z}_2\ (\mathcal{G}_{--}(z_1, z_2))^3 \qquad \Phi(z_1)\frac{\bar{D}_1^2}{\partial_1^2}\delta^8(z_1 - z_2) = \int d^8 z_3\ \Phi(z_3)\delta^8(z_3 - z_1)\frac{\bar{D}_3^2}{\partial_2^2}\delta^8(z_3 - z_2)$$

This integral can be formulated in terms of standard Feynman technique

Buchbinder, Kuzenko et al '93
Buchbinder, Kuzenko et al '94

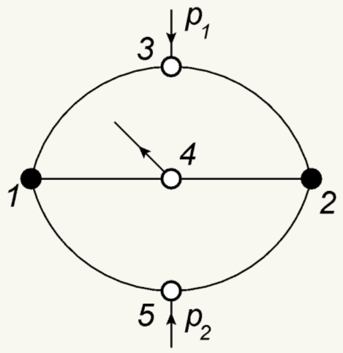
Fixing rules

We can derive rules for specific diagrams contributing to the chiral superpotential

1. As initial interaction is cubic, the correction should be proportional to cubic potential, i.e. every diagram must contain three second derivatives of initial chiral potential.
2. To satisfy the relation (*) the resulting integral up to last integration must contain number of squares of chiral covariant derivatives by one more than antichiral ones
3. Dimensional restriction: $2L + 1 - n_2 - n_3 = 0$

Only possible supergraph restricted by the rules at the two-loop level looks like

$\lim_{p_1, p_2 \to 0}$

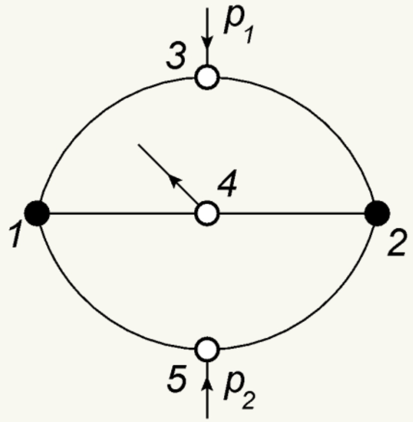


2-loop contribution

The first contribution to this potential is given by the two-loop diagram

$$\Gamma^{(2)}(\Phi) = \frac{\bar{\lambda}^2}{3! \times 2} \int d^6\bar{z}_1 d^6\bar{z}_2 \ (\mathcal{G}_{--}(z_1, z_2))^3$$

$$\Gamma^{(2),h}(\Phi) = \lambda \frac{|\lambda|^4}{3! \times 2} \int \prod_{i=1}^{5} d^8 z_i \Phi(z_3)\Phi(z_4)\Phi(z_5) \left\{ \frac{1}{\partial_1^2}\delta_{1,3} \frac{D_2^2 \bar{D}_3^2}{16\partial_2^2}\delta_{3,2} \quad \frac{1}{16\partial_2^2}\delta_{2,4} \frac{D_1^2 \bar{D}_4^2}{16\partial_1^2}\delta_{1,4} \frac{D_1^2 \bar{D}_5^2}{16\partial_1^2}\delta_{1,5} \frac{D_2^2}{4\partial_2^2}\delta_{2,5} \right\}$$



...after D-algebra routine:

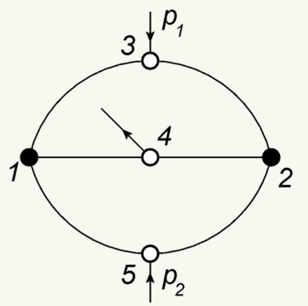
$$\lim_{p_1, p_2 \to 0} I_2(p_1, p_2) = \int \frac{d^4 q_1}{(4\pi)^4} \frac{d^4 q_2}{(4\pi)^4} \frac{q_1^2 p_1^2 + q_2^2 p_2^2 - 2(q_1 \cdot q_2)(p_1 \cdot p_2)}{q_1^2 q_2^2 (q_1 + q_2)^2 (q_1 - p_1)^2 (q_2 - p_2)^2 (q_1 + q_2 - p_1 - p_2)^2}$$

Integral easily can be evaluated by IBP and uniqueness relation

2-loop contribution

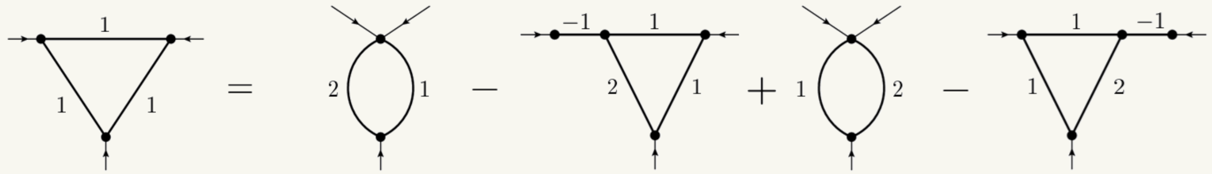
Buchbinder, Kuzenko'94
Buchbinder, Petrov et al '94
Jack, Jones, West'86

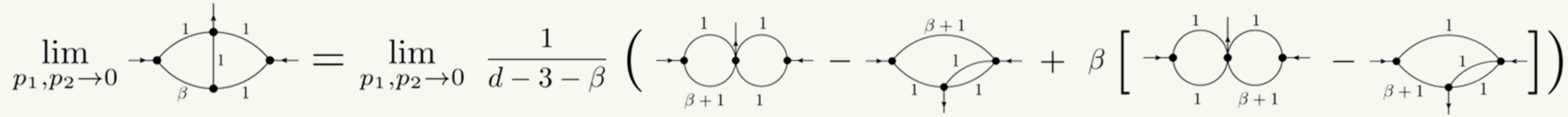
$$\lim_{p_1,p_2\to 0} I_2(p_1,p_2) = \int \frac{d^4q_1}{(4\pi)^4}\frac{d^4q_2}{(4\pi)^4}\frac{q_1^2p_1^2+q_2^2p_2^2-2(q_1\cdot q_2)(p_1\cdot p_2)}{q_1^2q_2^2(q_1+q_2)^2(q_1-p_1)^2(q_2-p_2)^2(q_1+q_2-p_1-p_2)^2}$$



Chetyrkin, Tkachev'81

IBP relation





Integral easily can be evaluated by IBP and uniqueness relation

Kazakov'84

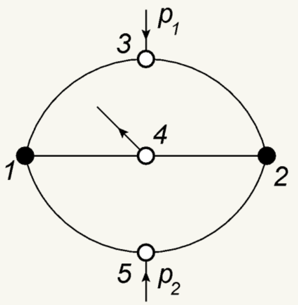
$$\mathbf{W}^{(2)}(\Phi) = \hbar^2\frac{|\lambda|^4}{(4\pi)^4}\frac{1}{2}\zeta(3)\ \lambda\Phi^3$$

The result is finite (no need to renormalise) and given as integral over the half-superspace!
Also it is not holomorphic by coupling constant!

Buchbinder, Kuzenko'98
Buchbinder, Petrov et al '94
Jack, Jones, West'86

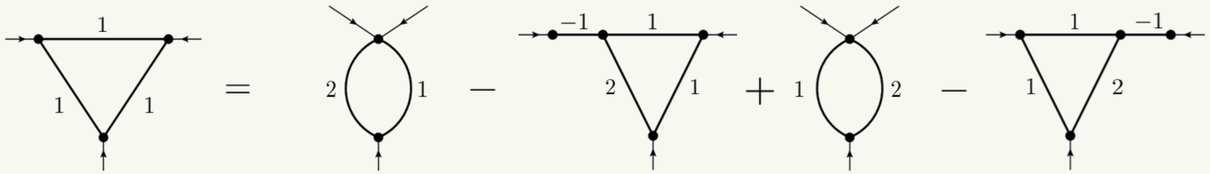
2-loop contribution

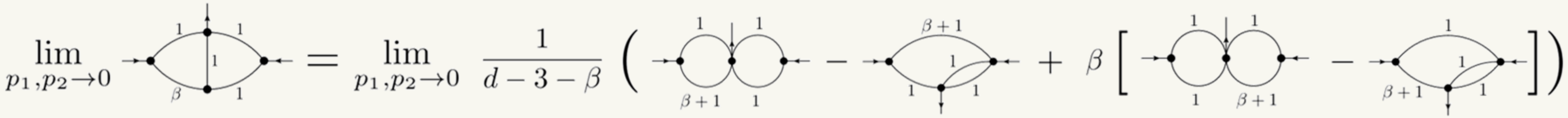
$$\lim_{p_1,p_2\to 0} \quad I_2(p_1,p_2) = \int \frac{d^4q_1}{(4\pi)^4}\frac{d^4q_2}{(4\pi)^4}\frac{q_1^2p_1^2+q_2^2p_2^2-2(q_1\cdot q_2)(p_1\cdot p_2)}{q_1^2q_2^2(q_1+q_2)^2(q_1-p_1)^2(q_2-p_2)^2(q_1+q_2-p_1-p_2)^2}$$



Chetyrkin, Tkachev'81

IBP relation





Integral easily can be evaluated by IBP and uniqueness relation

Kazakov'84

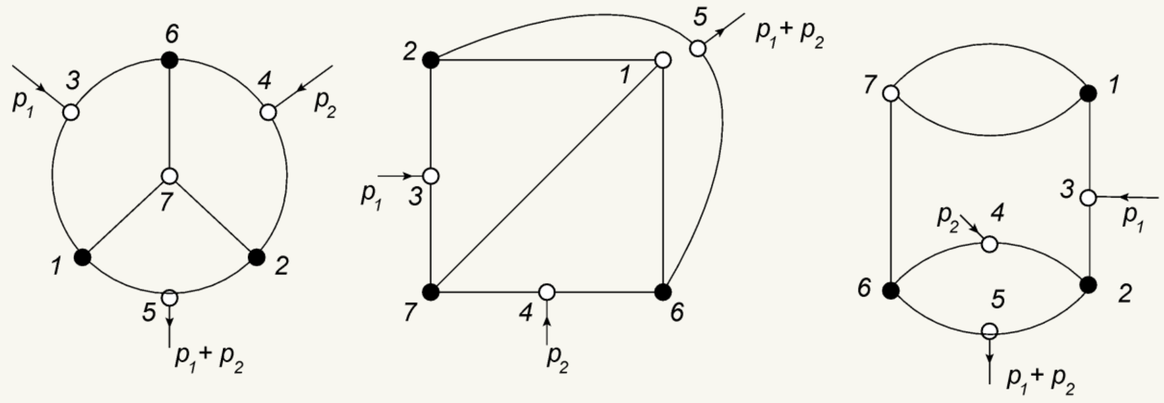
$$\mathbf{W}^{(2)}(\Phi) = \hbar^2\frac{|\lambda|^4}{(4\pi)^4}\frac{1}{2}\zeta(3)\ \lambda\Phi^3$$

Seiberg'93

We observe violation of Seiberg's holomorphy principle:
superpotential should be holomorphic in the fields and in the coupling constants but here it is not

3-loop contribution

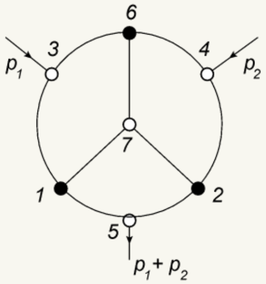
There are only three possible triple-loop diagrams contributing to CESP



a) b) c)

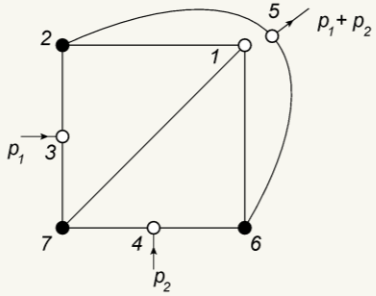
Two are finite and one is divergent due to presence of two-point integral (counterterms are needed)

Finite 3-loop contribution



a)

$$\Gamma^{(3a),h}(\Phi) = \lambda \frac{|\lambda|^6}{3!\times 8} \int \prod_{i=1}^{7} d^8 z_i \; \Phi(z_3)\Phi(z_4)\Phi(z_5)$$
$$\Big\{ \frac{1}{\partial_1^2}\delta_{1,3}\frac{\bar{D}_3^2 D_6^2}{16\partial_6^2}\delta_{3,6}\frac{D_6^2\bar{D}_5^2}{16\partial_6^2}\delta_{6,5}\frac{D_2^2}{4\partial_2^2}\delta_{5,2}$$
$$\frac{1}{\partial_2^2}\delta_{2,4}\frac{\bar{D}_4^2 D_1^2}{16\partial_1^2}\delta_{4,1}\frac{1}{\partial_1^2}$$
$$\delta_{1,7}\frac{\bar{D}_7^2 D_6^2}{16\partial_6^2}\delta_{7,6}\frac{\bar{D}_7^2 D_2^2}{16\partial_2^2}\delta_{7,2}\Big\}$$



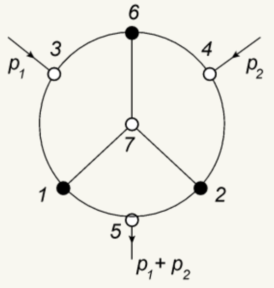
b)

$$\Gamma^{(3b),h}(\Phi) = \lambda \frac{|\lambda|^6}{3!\times 8} \int \prod_{i=1}^{7} d^8 z_i \; \Phi(z_3)\Phi(z_4)\Phi(z_5)$$
$$\Big\{ \frac{1}{\partial_2^2}\delta_{2,3}\frac{\bar{D}_3^2 D_7^2}{16\partial_7^2}\delta_{3,7}\frac{\bar{D}_7^2 D_4^2}{16\partial_7^2}\delta_{7,4}$$
$$\frac{1}{\partial_6^2}\delta_{4,6}\frac{D_6^2}{4\partial_6^2}\delta_{6,5}\frac{\bar{D}_5^2 D_2^2}{16\partial_2^2}$$
$$\delta_{5,2}\frac{D_2^2\bar{D}_1^2}{16\partial_2^2}\delta_{2,1}\frac{\bar{D}_1^2 D_6^2}{16\partial_6^2}\delta_{1,6}\frac{1}{\partial_7^2}\delta_{1,7}\Big\}$$

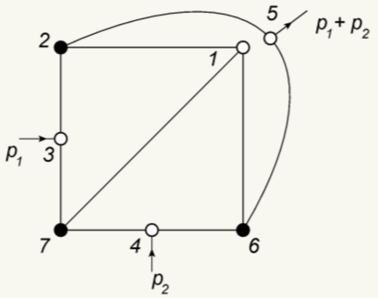
…D-algebra routine…

(`SusyMath.m` and other packages)

Finite 3-loop contribution



a)



b)

$$I_3^{(a)}(p_1,p_2) = \lim_{p_{1,2}\to 0}\int\prod_{i=1}^{3}\frac{d^4q_i}{(4\pi)^4}\,\frac{(a_1p_1^2+a_2p_2^2+a_3(p_1\cdot p_2))}{q_1^2(q_1+p_1)^2(p_1+q_2)^2(q_2+p_1+p_2)^2}\,\frac{1}{(q_2+p_1+p_2)(q_1-q_2)^2q_3^2(q_1-q_3)^2(q_3-q_2)^2}$$

$$a_1 = q_1^2\,(q_2-q_3)^2$$

$$a_2 = q_2^2\,(q_1-q_3)^2$$

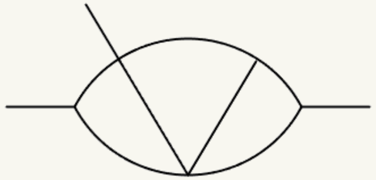
$$a_3 = -2\left(q_3^2(q_1\cdot q_2) - q_2^2(q_3\cdot q_1) + q_1^2(q_2\cdot q_3)\right)$$

$$I_3^{(b)}(p_1,p_2) = \lim_{p_1,p_2\to 0}\int\prod_{i=1}^{3}\frac{d^4q_i}{(4\pi)^4}\,\frac{(b_1p_1^2+b_2p_2^2+b_3(p_1\cdot p_2))}{(q_1-p_1)^2q_1^2q_2^2(q_2+p_2)^2(q_3+p_2)}\,\frac{1}{(q_3-p_1)^2(q_3-q_1)^2(q_3-q_2)^2(q_1-q_2)^2}$$

$$b_1 = q_2^2\,(q_1-q_3)^2$$

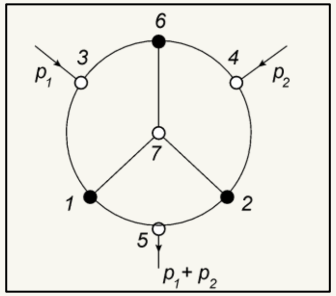
$$b_2 = q_1^2\,(q_2-q_3)^2$$

$$b_3 = -2\left(q_1^2\,(q_2-q_3)^2 + q_3^2\,(q_1-q_2)^2 - q_2^2\,(q_1-q_3)^2\right)$$

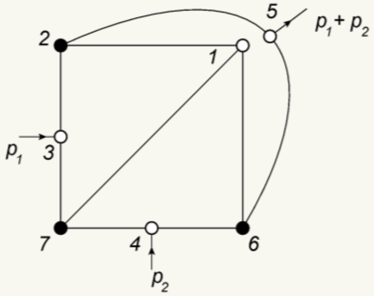


All these integrals reduces to a simple topology

Finite 3-loop contribution

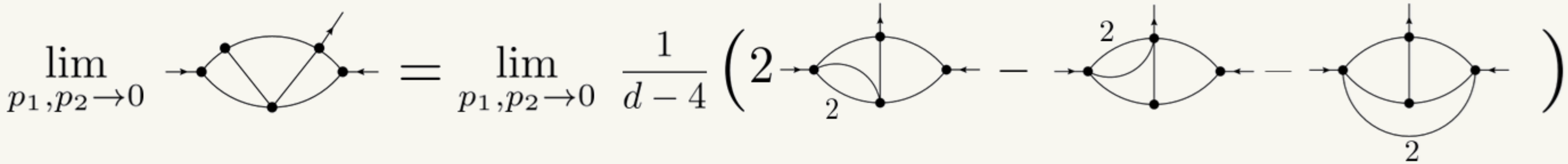


a)



b)

Integral easily can be evaluated by IBP and uniqueness relation



D-algebra and evaluation of master-integral

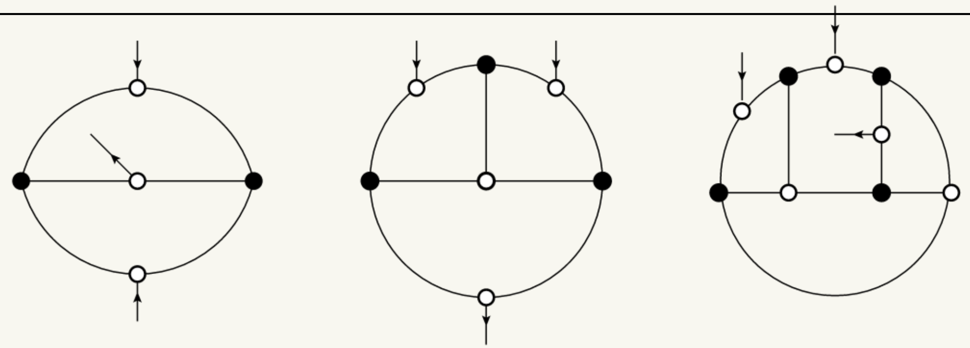
$$\Gamma^{(3a),h}[\Phi] = |\lambda|^6 \frac{5}{12}\zeta(5) \int d^6z \ \lambda\Phi^3(z)$$

D-algebra and evaluation of master-integral

$$\Gamma^{(3b),h}[\Phi] = |\lambda|^6 \frac{5}{12}\zeta(5) \int d^6z \ \lambda\Phi^3(z)$$

The 3-loop finite contribution

brick-wall type
supegraphs:



$+O(\lambda^{11})$

Distinguished series of diagrams. If there were no non-planar and divergent graphs, one could use the apparatus of superfishnet models/chiral part of beta deformed $\mathcal{N}$=1 SYM

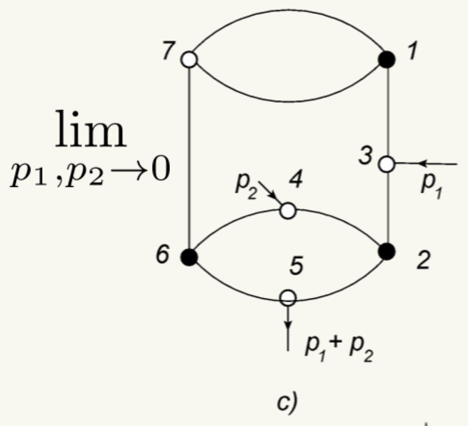
Kade, Staudacher'24
Broadhurst,Kreimer'96
Argurio, Feretti'03
Seiberg'93

Broadhurst-Kreimer-type relation

$$\sim \sum_L (-1)^L c_L \frac{\zeta(2L-1)}{(L-2)!}$$

- In non-perturbative approach there are no violation of Seiberg's principle
- 1PI action is a special case

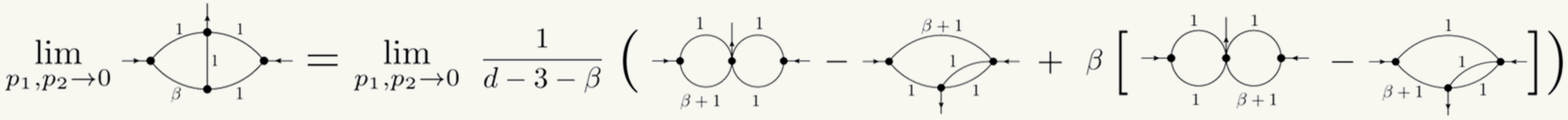
Divergent 3-loop contribution



$$\Gamma^{(3c),h}(\Phi) = \lambda \frac{|\lambda|^6}{3! \times 8} \int \prod_{i=1}^{7} d^8 z_i \ \Phi(z_3)\Phi(z_4)\Phi(z_5)$$
$$\Big\{ \frac{1}{\partial_1^2} \delta_{1,3} \frac{\bar{D}_3^2 D_2^2}{16\partial_2^2} \delta_{3,2} \frac{\bar{D}_4^2 D_2^2}{16\partial_2^2} \delta_{2,4} \frac{D_6^2}{4\partial_6^2} \delta_{4,6}$$
$$\frac{\bar{D}_5^2 D_6^2}{16\partial_6^2} \delta_{6,5} \frac{1}{\partial_2^2}$$
$$\delta_{5,2} \frac{1}{\partial_6^2} \delta_{6,7} \frac{\bar{D}_7^2 D_1^2}{16\partial_1^2} \delta_{1,7} \frac{\bar{D}_7^2 D_2^1}{16\partial_1^2} \delta_{7,1} \Big\}$$

$$\mathbf{K}_{1,div} = \frac{|\lambda|^2}{2\epsilon} \Phi\bar{\Phi}$$

$$\Gamma^{(3c),h}[\Phi] = \left(-\frac{3}{8\epsilon}\zeta(3) + O(\epsilon^0) \right) |\lambda|^6 \int d^6 z \ \lambda \Phi^3.$$



Final contribution

$$\Gamma^{(3c),h}[\Phi] = -|\lambda|^6 \left(\frac{3\zeta(3)}{8\epsilon} + \frac{3}{2}\zeta(3) + \frac{9}{16}\zeta(4) \right) \int d^6 z \ \lambda \Phi^3(z)$$

Final result and prospects

Result of three-loop contribution:

$$\bar{\Gamma}^{(3)}_{chiral}[\Phi] = \hbar^3 \int d^6z \, \mathbf{W}^{(3)}_{div}(\Phi) + \mathbf{W}^{(3)}_{fin}(\Phi),$$

total contribution

$$\mathbf{W}^{(3)}_{div} = -\frac{3\lambda|\lambda|^6}{8(4\pi)^8\epsilon}\Phi^3(z)$$

divergent contribution

$$\mathbf{W}^{(3)}_{fin} = -\frac{\lambda|\lambda|^6}{(4\pi)^8}\left(\frac{3}{2}\zeta(3) + \frac{9}{16}\zeta(4) - \frac{5}{6}\zeta(5)\right)\Phi^3(z)$$

finite contribution

Kade, Staudacher'24

- Four-loop calculations?
- Does $\mathcal{N}$=2 case contain such a non-holomorphic corrections in 1PI effective potential?
- General $\mathcal{N}$=1 chiral supersymmetric model three-loop?
- $\mathcal{N}$=1 beta deformed SYM (superfishnet model)/other SYM models?

Thank you for attention!