"Metric-affine-like" generalization of YM

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Motivation

	Similarities	Differences
YM	$\nabla_a \boldsymbol{\varphi} = (\partial_a - ie\boldsymbol{A}_a)\boldsymbol{\varphi}$, where $\boldsymbol{\varphi} \cong \varphi^\alpha$, $\boldsymbol{A}_a \cong A_{a\alpha}{}^\beta$, $\boldsymbol{F}_{ab} = \partial_a \boldsymbol{A}_b - \partial_b \boldsymbol{A}_a - ie\left[\boldsymbol{A}_a, \boldsymbol{A}_b\right]$.	DoFs in $\boldsymbol{A}_a$, $S_{\text{YM}} = -\frac{1}{4}\int d^d x\,\sqrt{g}\,\text{tr}\left(\boldsymbol{F}_{ab}\boldsymbol{F}^{ab}\right)$.
EG	$\nabla_a v^b = \partial_a v^b + \Gamma_{ac}{}^b v^c$, $R_{abc}{}^d = \partial_a \Gamma_{bc}{}^d - \partial_b \Gamma_{ac}{}^d$ $+ \Gamma_{ah}{}^d \Gamma_{bc}{}^h - \Gamma_{bh}{}^d \Gamma_{ac}{}^h$.	DoFs not in $\Gamma_{ac}{}^b$, but in g_{ab}, $S_{\text{HE}} = \frac{M_P^2}{2}\int d^d x\,\sqrt{g}\,R$.

- The Levi-Civita connection is defined by the torsion-free and covariant constancy of metric conditions:
  $T_{ab}{}^c = 0, \quad \nabla_a g_{bc} = 0 \quad \Rightarrow \quad \Gamma_{ac}{}^b = \frac{1}{2} g^{bd}\left(\partial_a g_{cd} + \partial_c g_{ad} - \partial_d g_{ac}\right)$.
- If we want to make the two theories even more similar, we should treat the connection ∇_a and the metric g_{ab} as two independent variables, i.e. $\nabla_a g_{bc} \neq 0$. This approach is well and long known — metric-affine gravity (MAG).
- But we are now interested in a simpler case: how to construct a "metric-affine-like" generalization of YM? Who is the "partner" of the potential $\boldsymbol{A}_a$ in this case?

Who is the “partner” of the potential $\boldsymbol{A}_a$?

Hermitian form

For definiteness, we will consider $U(n)$ throughout. Then the stucture in fibers is the form $g_{\alpha\beta'}$, which is:

- Hermitian $\bar{g}_{\alpha\beta'} = g_{\alpha\beta'}$ (analogous to the symmetry of the metric),
- non-degenerate $g_{\alpha\beta'} g^{\beta\beta'} = \delta_\alpha^\beta$ It allows us to raise and lower indices, with primed ones becoming unprimed and vice versa, e.g., $\bar{\varphi}_\alpha = g_{\alpha\beta'}\bar{\varphi}^{\beta'}$.

[Primed and unprimed color indexes are the direct analogue of undotted and dotted indices in Weyl 2-spinors.]

An example—a charged scalar

$$
\begin{aligned}
|\varphi|^2 &= g_{\alpha\beta'}\varphi^\alpha\bar{\varphi}^{\beta'}, \\
\mathcal{L}_\varphi &= -\frac{1}{2} g_{\alpha\beta'} g^{ab} \nabla_a \varphi^\alpha \nabla_b \bar{\varphi}^{\beta'} - P\big(|\varphi|^2\big),
\end{aligned}
$$

where $P\big(|\varphi|^2\big)$ is a self-interaction potential.

Connection

Definition of potentials and curvatures

Let us define $\mathcal{A}_a[\tilde{\nabla} - \nabla] \cong \mathscr{A}_{a\alpha}{}^{\beta}$ and $\mathcal{F}_{ab}[\nabla] \cong \mathscr{F}_{ab\alpha}{}^{\beta}$ as

$$(\tilde{\nabla}_a - \nabla_a)\psi^\alpha = \mathscr{A}_{a\beta}{}^{\alpha}\psi^\beta, \qquad [\nabla_a, \nabla_b]\psi^\alpha = \mathscr{F}_{ab\beta}{}^{\alpha}[\nabla]\psi^\beta.$$

Curvatures transformations and Bianchi identities:

$$\mathcal{F}_{ab}[\tilde{\nabla}] - \mathcal{F}_{ab}[\nabla] = \nabla_a\mathcal{A}_b - \nabla_b\mathcal{A}_a + [\mathcal{A}_a, \mathcal{A}_b], \qquad \nabla_{[a}\mathcal{F}_{bc]} = 0.$$

Important! Except the anti-symmetry in the first pair of indices and the Bianchi identities, no additional conditions are imposed on $\mathcal{F}_{ab}[\nabla]$. It is not an anti-Hermitian (there is no such a concept without $g_{\alpha\beta'}$!), but an arbitrary complex matrix.

$GL(n, \mathbb{C})$ gauge symmetry

Let $\boldsymbol{u} \cong u_\alpha^\beta$ and $\boldsymbol{U} \cong U_\alpha^\beta$ be two arbitrary (not unitary!) mutually inverse matrices: $\boldsymbol{uU} = \boldsymbol{Uu} = \mathbf{1}$. Then

$$H^{\beta_1\ldots\beta_q}_{\alpha_1\ldots\alpha_p} \mapsto \tilde{H}^{\beta_1\ldots\beta_q}_{\alpha_1\ldots\alpha_p} = U^{\beta_1}_{\gamma_1}\cdots U^{\beta_q}_{\gamma_q}\ H^{\gamma_1\cdots\gamma_q}_{\delta_1\ldots\delta_p}\ u^{\delta_1}_{\alpha_1}\cdots u^{\delta_p}_{\alpha_p},$$

$$\mathcal{A}_a[\tilde{\nabla} - \nabla] = \boldsymbol{U}\nabla_a\boldsymbol{u}$$

is a symmetry of *of any* action.

Hermitian form $g_{\alpha\beta'}$ and YM-deviation vector $\boldsymbol{N}_a$

The connection ∇_a is real in the following sense:

$$\overline{\nabla_a \varphi^\alpha} = \nabla_a \bar{\varphi}^{\alpha'} \quad \Rightarrow \quad (\tilde{\nabla}_a - \nabla_a)\bar{\varphi}^{\alpha'} = \bar{\mathscr{A}}_{a\beta'}{}^{\alpha'} \bar{\varphi}^{\beta'}.$$

Hermitian conjugation

$$\boldsymbol{M}^\dagger \cong \bar{M}_\alpha^\beta = g_{\alpha\alpha'} g^{\beta\beta'} \bar{M}_{\beta'}^{\alpha'}.$$

Split into Hermitian and anti-Hermitian parts:

$$\mathcal{A}_a = \boldsymbol{B}_a - i\boldsymbol{A}_a, \qquad \boldsymbol{B}_a = \mathsf{Hrm}\,\mathcal{A}_a = \frac{1}{2}(\mathcal{A}_a + \mathcal{A}_a^\dagger), \qquad \boldsymbol{A}_a = \mathsf{aHrm}\,\mathcal{A}_a = \frac{i}{2}(\mathcal{A}_a - \mathcal{A}_a^\dagger),$$

$$\mathcal{F}_{ab} = \boldsymbol{G}_{ab} - i\boldsymbol{F}_{ab}, \quad \boldsymbol{G}_{ab} = \mathsf{Hrm}\,\mathcal{F}_{ab} = \frac{1}{2}(\mathcal{F}_{ab} + \mathcal{F}_{ab}^\dagger), \quad \boldsymbol{F}_{ab} = \mathsf{aHrm}\,\mathcal{F}_{ab} = \frac{i}{2}(\mathcal{F}_{ab} - \mathcal{F}_{ab}^\dagger).$$

Definition of YM-deviation vector

$$\boldsymbol{N}_a \cong N_a{}_\alpha^\beta = -\frac{1}{2} g^{\beta\beta'} \nabla_a g_{\alpha\beta'}.$$

This is a Hermitian vector—the analogue of non-metricity in MAG.

Consequences of $\boldsymbol{N}_a \neq 0$

Noncommutativity of Hermitian conjugation and derivative

$$\nabla_a\left(\boldsymbol{M}^\dagger\right) = (\nabla \boldsymbol{M})^\dagger + 2\left[\boldsymbol{N}_a, \boldsymbol{M}^\dagger\right].$$

The key relation

The Hermitian part of the curvature of $\boldsymbol{G}_{ab}$ is completely expressed in terms of the YM-deviation vector $\boldsymbol{N}_a$:

$$\boldsymbol{G}_{ab} = \nabla_a \boldsymbol{N}_b - \nabla_b \boldsymbol{N}_a - 2\left[\boldsymbol{N}_a, \boldsymbol{N}_b\right].$$

Proof: $[\nabla_a, \nabla_b] g_{\alpha\beta'} = -\mathscr{F}_{ab}{}^{\gamma}_{\alpha} g_{\gamma\beta'} - \bar{\mathscr{F}}_{ab}{}^{\gamma'}_{\beta'} g_{\alpha\gamma'} = -2G_{ab\alpha\beta'} = -2(\nabla_a N_{b\alpha\beta'} - \nabla_b N_{a\alpha\beta'})$.

Hermitian form transformations

Let us define

$$\boldsymbol{\omega} \cong \omega^{\beta}_{\alpha} = \tilde{g}_{\alpha\beta'} g^{\beta\beta'}, \qquad \boldsymbol{\Omega} \cong \Omega^{\beta}_{\alpha} = g_{\alpha\beta'} \tilde{g}^{\beta\beta'}, \qquad \tilde{g}_{\alpha\beta'} = \omega^{\beta}_{\alpha} g_{\beta\beta'}.$$

It is easy to show that these matrices are Hermitian and mutually inverse

$$\boldsymbol{\omega}^\dagger = \boldsymbol{\omega}, \qquad \boldsymbol{\Omega}^\dagger = \boldsymbol{\Omega}, \qquad \boldsymbol{\omega}\boldsymbol{\Omega} = \boldsymbol{\Omega}\boldsymbol{\omega} = \mathbf{1}.$$

Different transformations of fields

Transformations of $\boldsymbol{N}_a$

$$\boldsymbol{N}_a[\tilde{\nabla},\tilde{g}] = \boldsymbol{\Omega}\boldsymbol{N}_a[\nabla,g]\boldsymbol{\omega} - \frac{1}{2}\boldsymbol{\Omega}\nabla_a\boldsymbol{\omega} + \frac{1}{2}\left(\mathcal{A}_a + \boldsymbol{\Omega}\mathcal{A}_a^\dagger\boldsymbol{\omega}\right),$$

$$\delta_g\boldsymbol{N}_a = -\frac{1}{2}\nabla_a\boldsymbol{h} + [\boldsymbol{N}_a,\boldsymbol{h}], \qquad \delta_B\boldsymbol{N}_a = \boldsymbol{B}_a, \qquad \delta_A\boldsymbol{N}_a = 0.$$

Transformations of $\boldsymbol{F}_{ab}$ and $\boldsymbol{G}_{ab}$ with Hermitian form

$\mathcal{F}_{ab}[\nabla]$ does not depend on $g_{\alpha\beta'}$ at all, $\mathcal{F}^\dagger_{ab}[\nabla,\tilde{g}] = \boldsymbol{\Omega}\mathcal{F}^\dagger_{ab}[\nabla,g]\boldsymbol{\omega}$.

Transformations of $\boldsymbol{F}_{ab}$ and $\boldsymbol{G}_{ab}$ with connection

$$\boldsymbol{G}_{ab}[\tilde{\nabla}] - \boldsymbol{G}_{ab}[\nabla] = \check{\boldsymbol{D}}_{ab} + i\check{\boldsymbol{K}}_{ab} - \hat{\boldsymbol{K}}_{ab} - \boldsymbol{C}_{ab},$$

$$\boldsymbol{F}_{ab}[\tilde{\nabla}] - \boldsymbol{F}_{ab}[\nabla] = \hat{\boldsymbol{D}}_{ab} + i\hat{\boldsymbol{K}}_{ab} + \check{\boldsymbol{K}}_{ab} + \check{\boldsymbol{C}}_{ab} - \hat{\boldsymbol{C}}_{ab}.$$

Where we introduce auxiliary quantities:

$$\check{\boldsymbol{D}}_{ab} = \nabla_a\boldsymbol{B}_b - \nabla_b\boldsymbol{B}_a, \qquad \check{\boldsymbol{K}}_{ab} = i[\boldsymbol{N}_a,\boldsymbol{B}_b] - i[\boldsymbol{N}_b,\boldsymbol{B}_a],$$

$$\hat{\boldsymbol{D}}_{ab} = \nabla_a\boldsymbol{A}_b - \nabla_b\boldsymbol{A}_a, \qquad \hat{\boldsymbol{K}}_{ab} = i[\boldsymbol{N}_a,\boldsymbol{A}_b] - i[\boldsymbol{N}_b,\boldsymbol{A}_a],$$

$$\check{\boldsymbol{C}}_{ab} = i[\boldsymbol{B}_a,\boldsymbol{B}_b], \qquad \hat{\boldsymbol{C}}_{ab} = i[\boldsymbol{A}_a,\boldsymbol{A}_b], \qquad \boldsymbol{C}_{ab} = i[\boldsymbol{A}_a,\boldsymbol{B}_b] - i[\boldsymbol{A}_b,\boldsymbol{B}_a].$$

$GL(n,\mathbb{C}) \to U(n)$ spontaneous symmetry breaking

Gauge transformations of the Hermitian form

$GL(n,\mathbb{C})$ gauge transformations, generally speaking, change the Hermitian form:

$$g_{\alpha\alpha'} \mapsto \tilde{g}_{\alpha\alpha'} = u^{\beta}_{\alpha}\bar{u}^{\beta'}_{\alpha'}g_{\beta\beta'} \quad \Rightarrow \quad \boldsymbol{\omega} = \boldsymbol{u}^{\dagger}\boldsymbol{u}, \quad \boldsymbol{\Omega} = \boldsymbol{U}\boldsymbol{U}^{\dagger}.$$

Hence, $g_{\alpha\beta'}$ does not change if the transformations are unitary $\boldsymbol{U} = \boldsymbol{u}^{\dagger}$.

Infinitesimal transformations $\boldsymbol{u} = \mathbf{1} + \boldsymbol{\epsilon}$, $\boldsymbol{\omega} = \mathbf{1} + \boldsymbol{h}$:

$$\boldsymbol{\epsilon} = \boldsymbol{\beta} - i\boldsymbol{\alpha}, \quad \boldsymbol{\beta} = \mathsf{Hrm}\,\boldsymbol{\epsilon}, \quad \boldsymbol{\alpha} = \mathsf{aHrm}\,\boldsymbol{\epsilon} \quad \Rightarrow \quad \boldsymbol{h} = 2\boldsymbol{\beta}.$$

$$\boldsymbol{A}_a = \nabla_a\boldsymbol{\alpha} - [\boldsymbol{N}_a, \boldsymbol{\alpha}] + i[\boldsymbol{N}_a, \boldsymbol{\beta}], \quad \boldsymbol{B}_a = \nabla_a\boldsymbol{\beta} - [\boldsymbol{N}_a, \boldsymbol{\beta}] - i[\boldsymbol{N}_a, \boldsymbol{\alpha}].$$

In this case, all matrices are transformed simply by similarity transformations:

$$\delta\boldsymbol{N}_a = [\boldsymbol{N}_a, \boldsymbol{\epsilon}], \qquad \delta\boldsymbol{F}_{ab} = [\boldsymbol{F}_{ab}, \boldsymbol{\epsilon}], \qquad \delta\boldsymbol{G}_{ab} = [\boldsymbol{G}_{ab}, \boldsymbol{\epsilon}].$$

Note that if $\boldsymbol{N}_a \neq 0$ or $\boldsymbol{G}_{ab} \neq 0$ they cannot be removed by gauge transformations.

So $g_{\alpha\beta'}$ is a "Higgs field", breaking $GL(n,\mathbb{C})$ to $U(n)$. And $\boldsymbol{h}$ is a "Goldstone boson" (and also a compensator or Stückelberg field for $\boldsymbol{B}_a$).

Field sources and Noether identities

Field sources:

$$\boldsymbol{\Lambda}^a = -\frac{\delta S}{\delta \boldsymbol{B}_a}, \qquad \boldsymbol{J}^a = \frac{\delta S}{\delta \boldsymbol{A}_a}, \qquad \boldsymbol{E} = -2\frac{\delta S}{\delta \boldsymbol{h}}.$$

Charged scalar

$$\mathcal{L}_\varphi = -\frac{1}{2} g_{\alpha\beta'} \nabla_a \varphi^\alpha \nabla^a \bar{\varphi}^{\beta'} - P\big(|\varphi|^2\big) \quad \Rightarrow \quad \boldsymbol{E} \cong E^\beta_\alpha = g_{\alpha\beta'} \nabla_a \varphi^\beta \nabla^a \bar{\varphi}^{\beta'} + 2P' \bar{\varphi}_\alpha \varphi^\beta,$$

$$\boldsymbol{J}_a \cong J_{a\alpha}{}^\beta = \frac{i}{2} g_{\alpha\beta'} \left(\varphi^\beta \nabla_a \bar{\varphi}^{\beta'} - \bar{\varphi}^{\beta'} \nabla_a \varphi^\beta\right), \qquad \boldsymbol{\Lambda}_a \cong \Lambda_{a\alpha}{}^\beta = \frac{1}{2} g_{\alpha\beta'} \nabla_a (\bar{\varphi}^{\beta'} \varphi^\beta).$$

Noether identities (pure mal-YM without matter):

If the theory has a gauge symmetry, the sources are not independent, but are related by Noether identities.

$$\nabla_a \boldsymbol{J}^a - [\boldsymbol{N}_a, \boldsymbol{J}^a] + i[\boldsymbol{N}_a, \boldsymbol{\Lambda}^a] = 0,$$
$$\nabla_a \boldsymbol{\Lambda}^a - [\boldsymbol{N}_a, \boldsymbol{\Lambda}^a] - i[\boldsymbol{N}_a, \boldsymbol{J}^a] = \boldsymbol{E}.$$

The action and EoMs

The action

$$\mathcal{L}_{\text{malYM}} = \frac{1}{e^2}\mathcal{L}_{F^2} + \frac{1}{\tilde{e}^2}\mathcal{L}_{G^2} + \frac{M^2}{\tilde{e}^2}\mathcal{L}_{N^2},$$

$$\mathcal{L}_{F^2} = -\frac{1}{4}\,\mathrm{tr}\left(\boldsymbol{F}_{ab}\boldsymbol{F}^{ab}\right), \qquad \mathcal{L}_{G^2} = -\frac{1}{4}\,\mathrm{tr}\left(\boldsymbol{G}_{ab}\boldsymbol{G}^{ab}\right), \qquad \mathcal{L}_{N^2} = -\frac{1}{2}\,\mathrm{tr}\left(\boldsymbol{N}_a\boldsymbol{N}^a\right).$$

[Of course, other terms can be introduced into the action (for example, $\mathrm{tr}(\boldsymbol{F}_{ab}\boldsymbol{G}^{ab})$, $\mathrm{tr}(\boldsymbol{F}_{ab}[\boldsymbol{N}^a, \boldsymbol{N}^b]$, etc.).]

EoMs for background fields

$$\nabla^b\boldsymbol{F}_{ab} - [\boldsymbol{N}^b, \boldsymbol{F}_{ab}] - i\frac{e^2}{\tilde{e}^2}[\boldsymbol{N}^b, \boldsymbol{G}_{ab}] = e^2\boldsymbol{J}_a^{\text{ext}},$$

$$\nabla^b\boldsymbol{G}_{ab} - [\boldsymbol{N}^b, \boldsymbol{G}_{ab}] + i\frac{\tilde{e}^2}{e^2}[\boldsymbol{N}^b, \boldsymbol{F}_{ab}] + M^2\boldsymbol{N}_a = -\tilde{e}^2\boldsymbol{\Lambda}_a^{\text{ext}},$$

$$M^2\nabla_a\boldsymbol{N}^a + i[\boldsymbol{G}_{ab}, \boldsymbol{F}^{ab}] = -\tilde{e}^2\boldsymbol{E}^{\text{ext}}.$$

Linearized equations and $\boldsymbol{h} = 0$ gauge

For small perturbations:

On a trivial background $\boldsymbol{N}_a = 0$, $\boldsymbol{G}_{ab} = \boldsymbol{F}_{ab} = 0$ (here $\Box = -g^{ab}\nabla_a\nabla_b$):

$$(\delta_a^b \Box + \nabla^b \nabla_a)\boldsymbol{A}_b = 0,$$

$$(\delta_a^b \Box + \nabla^b \nabla_a)\boldsymbol{B}_b + M^2(\boldsymbol{B}_a - \frac{1}{2}\nabla_a \boldsymbol{h}) = 0,$$

$$\Box \boldsymbol{h} + 2\nabla_a \boldsymbol{B}^a = 0.$$

$\boldsymbol{h} = 0$ gauge

We can always use spontaneously broken gauge symmetry to completely eliminate Goldstone bosons by redefining $\boldsymbol{B}_a \mapsto \boldsymbol{B}_a - \nabla_a \boldsymbol{h}/M$. This leads to the equations:

$$(\delta_a^b \Box + \nabla^b \nabla_a)\boldsymbol{A}_b = 0, \qquad (\delta_a^b(\Box + M^2) + \nabla^b \nabla_a)\boldsymbol{B}_b = 0,$$

i.e. to the massless field $\boldsymbol{A}_a$ + the massive Proca field $\boldsymbol{B}_a$.
[But in this gauge we have a problem with the non-decreasing propagator

$$G_a^b(\boldsymbol{k}) = \frac{1}{k^2 + M^2}\left(\delta_a^b + \frac{k_a k^b}{M^2}\right).]$$

Summary of main claims and results

- In the standard Yang-Mills theory, it is always implicitly assumed that the structure in the fibers is covariantly constant $\nabla_a g_{\alpha\beta'} = 0$.
- Accordingly, the "metric-affine-like" generalization of YM consists in dropping this condition $\nabla_a g_{\alpha\beta'} \neq 0$. Then the connection ∇_a and the Hermitian form $g_{\alpha\beta'}$ act as two independent variables.
- Any geometrically defined theory always has a general $GL(n, \mathbb{C})$ gauge symmetry. The Hermitian form $g_{\alpha\beta'}$ plays the role of a "Higgs field", spontaneously breaking this symmetry to $U(n)$.
- If the connection respects the structure in fibers, the potential and the curvature take values in the corresponding Lie algebra. In our case this is not so, then along with the usual Yang-Mills fields $\boldsymbol{A}_a$ and $\boldsymbol{F}_{ab}$, they have new Hermitian parts $\boldsymbol{B}_a$ and $\boldsymbol{G}_{ab}$.
- The fields $\boldsymbol{A}_a$ and $\boldsymbol{B}_a$ interact non-trivially. The field $\boldsymbol{A}_a$ is massless, and the field $\boldsymbol{B}_a$ can be given a mass M. The limit $M \to \infty$ restores the usual YM.