Lagrangian formulation of infinite- spin field in curved spacetime

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Aim: Brief review of Lagrangian formulation for infinite-spin field in curved spacetime within BRST approach for higher spin theories. Continuation of previous talk by S. Fedoruk.

Based on series of papers published during last year in collaboration with S. Fedoruk, A. Isaev, V. Krykhtin, M. Podoinitsyn.

Canonical BRST charge (BFV charge)

A distinctive feature of gauge theories is the presence of first-class constraints in phase space. Such constraints appear automatically when we move from the Lagrangian formulation of gauge theories to their Hamiltonian formulation.

Let (q^a, p_a) are the phase space coordinates, $H(q, p)$ is a Hamiltonian and $T_a(q, p)$ are the constraints. The constraints are called first-class if they satisfy the following relation in terms of Poisson brackets

$$\{T_a, T_b\} = C^c{}_{ab}(q, p) T_c$$

The functions $C^c{}_{ab}$ are determined by gauge algebra in Hamiltonian formalism. In this case, the action in the Hamiltonian formalism is written in the form

$$S_H[q, p] = \int dt \, (p_a \dot{q}^a - H(q, p) - \lambda^a T_a),$$

where λ^a are the Lagrange multipliers. This relation is derived when we move from Lagrangian formulation to Hamiltonian formulation. However, in principle, if we somehow define the constraints, their first class algebra and the action $S_H[q, p]$, then one can construct a Lagrangian gauge formulation corresponding to given Hamiltonian formulation with constraints.

Canonical BRST charge (BFV charge)

Let T_a the first-class constraints in the phase space of bosonic variable and let the functions $C^c{}_{ab}$ are the bosonic constants. We extend the phase space by anticommuting (fermionic) coordinates η^a and momenta $\mathcal{P}_a$ which are called the ghost variables. Define in the extended phase space the fermionic function $Q(q, p, \eta, \mathcal{P})$ by the rule

$$Q = \eta^a T_a + \frac{1}{2} C^a{}_{bc} \eta^b \eta^c \mathcal{P}_a.$$

One can show that Q is nilpotent in terms of graded Poisson brackets. This function is called BFV (Batalin-Fradkin-Vilkovisky) charge or canonical BRST (Becci-Ruet-Stora-Tyutin) charge.

After quantization, the BRST charge becomes an nilpotent operator acting in the state space of vectors $|\Psi\rangle, \quad |\Psi\rangle' = Q|\Psi\rangle$. This relation is invariant under transformation $|\Phi\rangle \to |\Phi\rangle + Q|\Lambda\rangle$. Last relation is realization of gauge invariance in state space.

Generic scheme of Lagrangian formulation for higher spin fields within the BRST approach

- The relations which describe the irreducible free field representations with given spin or helicity are interpreted as first class constraints of some unknown yet gauge theory in some Fock space.
- Using these constraints one constructs the Hermitian canonical BRST charge Q as an operator acting in extended Fock space.
- This BRST charge allows to obtain a real gauge invariant Lagrangian for free higher spin fields in terms of Fock space.
- The gauge invariant equations of motion for free higher spin fields have the form $Q|\Psi >= 0$ and reproduce, as their identical consequences, the relations defying the irreducible representations. This properties emphasizes a correctness of generic scheme.

Generic scheme of Lagrangian formulation for higher spin fields within the BRST approach

- Formulation in terms of ordinary fields actually requires only the calculation of vacuum averages of the product of a certain number of annihilation and creation operators.
- Higher spin field interactions are described as deformations of BRST charge with preservation of nilpotency.
- The formulation possesses extremely high gauge symmetry. Using the various gauge fixing conditions and eliminating part of auxiliary fields from equations of motion we can derive the various equivalent Lagrangian formulations.
- The scheme has been realized for bosonic and fermionic, massive and massless higher fields in Minkowski and AdS spaces of various dimensions.
- BRST approach always automatically leads to a gauge invariant Lagrangian formulation even for massive higher spin fields.

Lagrangian formulation for free infinite spin field theory in $4D$ Minkowski space

In this case the Lagrangian formulation is constructed as follows

1. Introduce an infinite collection of higher spin fields with two-component spinor indices $\varphi_{\alpha(s)}{}^{\dot\alpha(s)}(x)$
2. Introduce two sets of bosonic annihilation $a_\alpha, \bar a^{\dot\alpha}$ and creation $c^\alpha, \bar c_{\dot\alpha}$ operators with standard commutation relations
3. Introduce infinite sum of the Fock spaces of the vectors $|\varphi\rangle = \sum_{s=0}^{\infty} |\varphi_{(s)}\rangle$.
4. Introduce the operators $l_0, \quad l-\mu, \quad l^+-\mu$, where $l_0=\Box$ and $l=i(a\sigma^m\bar a)\partial_m$, and corresponding conjugate operator $l^\dagger-\mu$.

Lagrangian formulation for free infinite spin field theory in $4D$ Minkowski space

1. Construct the nilpotent BRST charge

$$Q = \eta_0 l_0 + \eta^+(l - \mu) + \eta(l^+ - \mu) + K\eta^+\eta\mathcal{P}_0,$$

   where $K = c^\alpha a_\alpha + \bar{c}_{\dot\alpha}\dot{a}^{\dot\alpha} + 2$.

2. Introduce the extended Fock space of the vectors

$$|\Phi\rangle = |\varphi\rangle + \eta_0\mathcal{P}^+|\varphi_1\rangle + \eta^+\mathcal{P}^+|\varphi_2\rangle.$$

3. Define the Lagrangian

$$\mathcal{L} = \int d\eta_0 \langle\Phi|Q|\Phi\rangle$$

   Lagrangian is invariant under the gauge transformation $|\Phi'\rangle = |\Phi\rangle + Q|\Lambda\rangle$.

One can show that this Lagrangian is an infinite sum of Lagrangians for free massless fields plus μ-dependent cross terms responsible for infinite spin field description. The formulation possesses extremely high gauge symmetry. Using the various gauge fixing conditions we can derive the various equivalent Lagrangian formulations.

Generalization for infinite spin fields in curved spacetime

Taking into account the Lagrangian for free infinite spin field theory, it is naturally to try to construct a theory in curved spacetime. First of all I want to describe a generic scheme and to announce the results.

- The the BRST construction is based on some first class constraint algebra which becames to be a gauge algebra of final Lagrangian theory. In the case of theory in Minkowski space, the constraints and their algebra was stipulated the conditions defying the irreducible representation of the Poincáre group. Since the Poincáre group is not a symmetry group of arbitrary curved space, we should find another way to go further.
- It is qualitative clear that if we want to preserve the concept of spin in a curved spacetime, we should restrict ourselves only by the space AdS, where the concept of spin exists.
- We will start with formal generalization of the infinite spin first class constraint algebra of flat space to arbitrary curved space assuming existence of the known flat limit of such an algebra. It is shown that condition of closure of generalized curved space algebra is realized only for AdS space.

Generalization for infinite spin fields in in curved spacetime

1. Minimal inclusion of interaction with gravity. We change in the constraints of flat theory the partial derivatives by covariant ones and then we check whether the algebra of new constraints is closed. The algebra is not closed!
2. We add non-minimal terms containing curvature and look for them from the algebra closure condition. This is not a trivial job at all. A priori there are no guarantees that this can be done. It is shown that the algebra of generalized constraints with appropriate non-minimal interaction is closed only on AdS space.
3. Taking into account this constraint algebra we apply the BRST formalism and derive the Lagrangian formulation. As in cases of infinite spin field theory in Minkowski space, this formulation has a large gauge freedom and by applying different gauges and/or eliminating part of auxiliary fields with help of equations of motion we can obtain the formally different but equivalent Lagrangians.

Generalization for infinite spin field in the AdS space.

Results

- Generalized constrains with non-minimal coupling to curvature

$$L_0 = l_0^{(cov)} + \frac{1}{2} r K^2 + 2\mu\, r^{1/2}\,,$$

$$L_1 = l_1^{(cov)} - \mu - \frac{1}{4}\, r^{1/2} K^2\,, \qquad L_1^+ = l_1^{(cov)} - \mu - \frac{1}{4}\, r^{1/2} K^2,$$

where $K = c^\alpha a_\alpha + \bar{c}_{\dot\alpha} \dot{a}^{\dot\alpha} + 2$ and parameter $r = -\kappa$ and $\kappa < 0$ is a scalar curvature of AdS space.

- Generalized first class constraint algebra

$$\begin{aligned}
[L_1, L_0] &= 0\,, \qquad [L_1^+, L_0] = 0\,, \\
[L_1^+, L_1] &= K L_0 + r^{1/2}(K+1) L_1 + r^{1/2}(K-1) L_1^+ = \\
&= K L_0 + \tfrac{1}{2} r^{1/2}(K+1) L_1 + \tfrac{1}{2} r^{1/2}(K-1) L_1^+ + \\
&\qquad + \tfrac{1}{2} r^{1/2} L_1^+ (K+1) + \tfrac{1}{2} r^{1/2} L_1 (K-1)
\end{aligned}$$

Generalization for infinite spin field in the *AdS* space.

- BRST charge

$$
\begin{aligned}
Q \; = \; & c_0(L_0 + r) + c_1^+ L_1 + c_1 L_1^+ + K c_1^+ c_1 b_0 + \\
& + \tfrac{1}{2} r^{1/2}(K+1)\, c_1^+ c_1 b_1 + \tfrac{1}{2} r^{1/2}(K-1)\, c_1^+ c_1 b_1^+ + \\
& + \tfrac{1}{2} r^{1/2}(K+1)\, b_1^+ c_1^+ c_1 + \tfrac{1}{2} r^{1/2}(K-1)\, b_1 c_1^+ c_1 \,, \\
& Q^2 = 0 \,, \qquad Q^+ = Q \,.
\end{aligned}
$$

- Lagrangian

$$
\begin{aligned}
\mathcal{L} \; = & \langle\varphi| \Big\{ (L_0 + r)|\varphi\rangle - \Big[L_1^+ + \tfrac{1}{2}\, r^{1/2}(K-1) \Big] |\varphi_1\rangle \Big\} \\
& - \langle\varphi_1| \Big\{ \Big[L_1 + \tfrac{1}{2}\, r^{1/2}(K-1) \Big] |\varphi\rangle - \Big[L_1^+ - \tfrac{1}{2}\, r^{1/2}(K+1) \Big] |\varphi_2\rangle + K|\varphi_1\rangle \Big\} \\
& - \langle\varphi_2| \Big\{ (L_0 + r)|\varphi_2\rangle - \Big[L_1 - \tfrac{1}{2}\, r^{1/2}(K+1) \Big] |\varphi_1\rangle \Big\}
\end{aligned}
$$

Generalization for infinite spin field in the AdS space.

- Gauge transformations (conditioned by nilpotency of BRST charge)

$$\delta|\varphi\rangle = \Big[L_1^+ + \tfrac{1}{2}\, r^{1/2}(K-1)\Big]|\lambda\rangle\,, \qquad \delta|\varphi_1\rangle = (L_0 + r)|\lambda\rangle\,,$$

$$\delta|\varphi_2\rangle = \Big[L_1 - \tfrac{1}{2}\, r^{1/2}(K+1)\Big]|\lambda\rangle\,.$$

- The Lagrangian and gauge transformations at $r = 0$ turn to Lagrangian and gauge transformations for bosonic infinite spin in Minkowski space.
- The Lagrangian and gauge transformations at $\mu = 0$ turn to Lagrangian and gauge transformations for infinite collection of bosonic higher spin fields in AdS space.
- Lagrangian formulation has universal triplet structure.
- Lagrangian formulation possesses very large gauge freedom. Partial gauges fixing and/or eliminating of part of auxiliary fields with help of equations of motion yields formally different but on-shell equivalent Lagrangian formulations.
- Lagrangian formulation in terms of component fields is obtained after calculating vacuum expectation values for products of annihilation and creation operators.

Thank you very much for attention.