# On the possibility of using the Majorana fermion in the quantum-field theory of elementary particles and as an effective field interacting with matter in models of statistical physics

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# Introduction

We consider a real relativistic fermion field with spin 1/2, proposed by Majorana as a model of an elementary particle in 1937. Although there are still no direct experimental confirmations of its existence, the Majorana fermion (MF) is still of interest to many researchers. In condensed matter physics, many models have been proposed in which MF is used as an effective field to describe various nontrivial phenomena. In this talk we propose an analysis of its features, following from the most general basic postulates of relativistic quantum theory. The possibilities of its inclusion in the modern Standard Model of the unified strong weak and electromagnetic interactions are also discussed, as well as its application to the construction of models in condensed matter physics.

We assume that the Majoràna fermion $\psi(x)$ has 4 components: $\psi(x) = \{\psi_1(x), ..., \psi_4(x)\}$, which are real functions of a point $x$ of the (3+1)-dimensional Minkowski space. In Lorentz transformations, $\psi(x)$ is transformed as a spinor, and the Lagrange function $L_0(\psi)$ in free theory is a bosonic, quadratic on $\psi(x)$, local, real scalar containing derivatives on the coordinates $x$ not higher than first order.
Thus, the action functional $S_0(\psi)$ in the most general case is written as

$$S_0(\psi) = \int L_0(\psi)dx = \int \sum_{k,j=1}^{4} \psi_j(x) \left( \sum_{\mu=0}^{3} \Gamma^{\mu}_{jk}\partial_\mu + M_{jk} \right) \psi_k(x)dx$$

Here the integration is over the entire Minkowski space, $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and $\Gamma^{\mu}_{jk}, M_{jk}$ are non-dependent on the $x$ coordinates. parameters.

# Formulation of model

In Majorana's proposed representation of gamma matrices

$$\gamma_0 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \gamma_1 = i \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \tag{1}$$

$$\gamma_2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \ \gamma_3 = i \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \tag{2}$$

where 0 denotes the $2 \times 2$-matrix with zero elements and $\sigma_1$, $\sigma_2$, $\sigma_3$ are Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

all matrices (1) and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ are purely imaginary. We will be more comfortable with the real ones $\bar{\gamma}^\mu = i\gamma^\mu$, $\mu = 0, 1, 2, 3$, $\bar{\gamma}^5 = i\gamma^5$. They satisfy the relations

$$\{\bar{\gamma}^\mu, \bar{\gamma}^\mu\} = -2g^{\mu\nu}, \ (\bar{\gamma}^0)^T = -\bar{\gamma}^0, \ (\bar{\gamma}^j)^T = \bar{\gamma}^j, \ j = 1, 2, 3.$$

We can represent the action functional $S_0(\psi)$ of the free theory as

$$S_0(\psi) = \psi M_v^{\mu}\partial_\mu\psi + \psi M_{sc}\psi.$$

It should be invariant with respect to Lorentz transformations. Therefore, $\psi M_v^{\mu}\psi$ is transformed as a vector and $\psi M_{sc}\psi$ is invariant by the Lorentz transformation of the spinor fields $\psi$. Since the fields $\psi$ are fermions, it that it holds: ${M_v^{\mu}}^T = M_v^{\mu}$, and ${M_{sc}^{\mu}}^T = -M_{sc}^{\mu}$. It follows from $\bar{\gamma}_0^T = -\bar{\gamma}_0^T$, $\bar{\gamma}_k^T = \bar{\gamma}_k$ by $k = 1, 2, 3$, $(\bar{\gamma}_0\bar{\gamma}_\alpha)^T = \bar{\gamma}_0\bar{\gamma}_\alpha$) by $\alpha = 0, 1, 2, 3$, ${\bar{\gamma}^5}^T = -\bar{\gamma}^5$, that the most general form of the Lorentz-invariant action functional for the Majorana field can be written as

$$S(\psi) = \psi(\bar{\gamma}^0(\bar{\gamma}^\mu\partial_\mu + m(\cos(\zeta) + \bar{\gamma}^5\sin(\zeta)))\psi. \qquad (3)$$

# Stationarity equation

The stationarity equation for the action $S(\psi)$ (3) has the form

$$\bar{\gamma}^0(\bar{\gamma}^\mu\partial_\mu + m(\cos(\zeta) + \bar{\gamma}^5\sin(\zeta))\psi = 0. \tag{4}$$

This is a linear homogeneous equation for the $\psi$ field. The condition for its solvability is that the determinant of the matrix of differential operator $L = \bar{\gamma}^0(\bar{\gamma}^\mu\partial_\mu + m(\cos(\zeta) + \bar{\gamma}^5\sin(\zeta))$ equals zero: $\det(L) = (\partial^2 + m^2)^2 = 0$, where $\partial^2 = \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2$. This means that each component of the spinor $\psi$ satisfies the equation

$$(\partial^2 + m^2)\psi = 0. \tag{5}$$

The rank of the system of equations (4) is two. Therefore, by solving two of them, we get the total solution.

From the first two equations (4) we find two linearly independent solutions in the form of

$$\psi_1 = \{(\partial_1 - m\sin(\zeta))\varphi_1, (m\cos(\zeta) - \partial_3)\varphi_1, (\partial_0 + \partial_2)\varphi_1, 0\},\} \quad (6)$$
$$\psi_2 = \{(\partial_3 + m\cos(\zeta))\varphi_2, (\partial_1 + m\sin(\zeta))\varphi_2, 0, -(\partial_0 + \partial_2)\varphi_2\}. \quad (7)$$

There in curly brackets are the four components of the spinors $\psi_1$ and $\psi_2$, expressed in terms of the independent real fermionic functions $\varphi_1(x), \varphi_2(x)$ satisfying equations (5) $L\varphi_1(x) = 0,\ L\varphi_2(x) = 0.$ Substating $\psi_1, \psi_2$ into (5) we obtain $\{0, 0, L\varphi_1, 0\} = 0$, and also $\{0, 0, 0, 0, L\varphi_2\} = 0$.

# Helicity

To describe the state of the $\psi$ field, we can use the helicity operator

$$\Sigma = \frac{\gamma_2\gamma_3\partial_1 - \gamma_1\gamma_3\partial_2 + \gamma_1\gamma_2\partial_3}{2\sqrt{\Delta}}, \quad \Delta = -\partial_1^2 - \partial_2^2 - \partial_3^2,$$

a as well as the projectors $P_{\Sigma\pm} = \mathbf{1}/2 \pm \Sigma$ onto a subspace with positive and negative helicity. Linear combinations $\psi_\pm = a_\pm\psi_1 + b_\pm\psi_2$ of spinors $\psi_1, \psi_2$, at

$$\begin{aligned}
a_+ &= \partial_0\partial_1 - \partial_2 m\sin(\zeta) - \sqrt{\Delta}(\partial_3 + m\cos(\zeta)),\\
b_+ &= \partial_0\partial_0\partial_3 + \partial_2 m\cos(\zeta) + \sqrt{\Delta}(\partial_1 - m\sin(\zeta)),\\
a_- &= \partial_0\partial_1 - \partial_2 m\sin(\zeta) + \sqrt{\Delta}(\partial_3 + m\cos(\zeta)),\\
b_- &= \partial_0\partial_3 + \partial_2 m\cos(\zeta) + \sqrt{\Delta}(m\sin(\zeta) - \partial_1)
\end{aligned}$$

are the eigenspinors of the helicity operator: $\Sigma\psi_\pm = \pm\frac{1}{2}\psi_\pm$.

# Propagator and currents

$\Delta = -\partial_1^2 - \partial_2^2 - \partial_3^2$ is a positive defined differential operator, and the spinors $\psi_1, \psi_2, \psi_+, \psi_-$ in coordinate representation and the helicity operator $\Sigma$ are real. Propagator of Majorana field has the form

$$(\partial^\mu \bar{\gamma}_\mu + m(\cos(\zeta) + \sin(\zeta)\bar{\gamma}_5))^{-1} =$$
$$= (\partial^\mu \bar{\gamma}_\mu + m(\cos(\zeta) - \sin(\zeta)\bar{\gamma}_5))(\partial_0^2 - \vec{\partial}^2 + m^2)^{-1}$$

The electric current vector $\psi\gamma^0\gamma^\mu\psi$ is zero due to the symmetry of all matrices $\gamma^0\gamma^\mu = (\gamma^0\gamma^\mu)^T$, $\mu = 0, 1, 2, 3$ in the Majorana representation (1) and fermion statistics of $\psi$.
The matrices $M_a^\mu = \gamma^0\gamma^5\gamma^\mu$ are antisymmetric: $M_a^{\mu\,T} = -M_a^\mu$, and the axial current $J_a^\mu(x) = \psi(x)M_a^\mu\psi(x)$ can be non-zero.

# Interactions of MF with fundamental fields

Now we consider the possibilities of constructing a functional describing the $\psi$-field interactions with fields of the Standard model. If we impose the requirement of locality and renormalizability of the model in space-time of dimension $d = 3 + 1$, then the interaction action cannot be a polynomial containing $\psi$ in degree $n > 2$. If the interaction does not violate Lorentz-invariance, and if we assume that the action of the interaction should be Hermite, the direct interaction of the MF with all spinor fields of the Standard Model appears to be inadmissible. As a result only scalar Higgs fields and gauge vector bosons remain as possible interaction partners of MF. Difficulties of realization of this possibility arise because of the gauge principle of interactions is fundamental for the Standard Model. It's something that we'd like one would like not to violate it by introducing into it the spinor fermion field $\psi$ without adding to it some additional properties, as a result of which $\psi$ loses its neutrality.

The action functional describing the interaction of the massless Majorana field $\psi$ with electromagnetic field $A^{\mu}$ is proposed as

$$S(A,\psi) = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \psi\bar{\gamma}^0(\hat{\partial} - \kappa\hat{A}_{\mu}\bar{\gamma}^5)\psi$$

where $F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$ and $\kappa$ is a massless coupling constant. It is invariant in respect to gauge transformations of the form

$$\psi(x) \to e^{\kappa\omega(x)\bar{\gamma}^5}\psi(x),\;\; A_{\mu}(x) \to A_{\mu}(x) + \partial_{\mu}\omega(x)$$

defined by scalar function $\omega(x)$.
For calculating the Green functions one needs to choose the gauge fixing. One con do it by using the action functional in the general Feynman gauge

$$S(A,\psi;\alpha) = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2\alpha}(\partial_{\mu}A^{\mu})^2 + \psi\bar{\gamma}^0(\hat{\partial} - \kappa\hat{A}\bar{\gamma}^5)\psi \quad (8)$$

with gauge defining parameter $\alpha$.

# Interaction of massles MF with electromagnetic field

The interaction of field $A^\mu, \psi$ is described in $S(A,\psi;\alpha)$ by the coupling matrix $V^\mu = \kappa\bar{\gamma}^0\bar{\gamma}^\mu\bar{\gamma}^5$. The purely photon Green functions are obtained from generating functional

$$G(J,\xi) = c\int e^{iS(A,\psi)+JA+\xi\psi}DAD\psi,\ (c)^{-1} = \int e^{iS(A,\psi)}DAD\psi$$

$G(J,\xi)$ with $\xi = 0$. The generating functional of them can be written as

$$G(J,0) = c'\int e^{iS'(A)+JA}DA,\ (c')^{-1} = \int e^{iS'(A)}DA,$$

$$S'(A) = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - Tr\ln(\mathbf{1} - \kappa D_\psi V_\mu A^\mu),$$

$$Tr\ln(\mathbf{1} - \kappa D_\psi V_\mu A^\mu) = \sum_{n=1}^{\infty}\frac{1}{2n}\kappa^{2n}Tr(D_\psi V_\mu A^\mu)^{2n}. \qquad (9)$$

Here, in the right hand side of (9), there are not terms with odd powers of $A$ and $G(J,0) = G(-J,0)$. It is analogously to Furry theorem of quantum electrodynamic (QED).

Doing the gauge transformation of the integration variables in expressions for $G(J,\xi)$ , we obtain the Ward identity for the generating functional of connected Green functions $W(J,\xi,\alpha) = \ln G(J,\xi,\alpha)$:

$$\left(\kappa\xi(x)\bar{\gamma}^5 \frac{\vec{\delta}}{\delta\xi(x)} - \frac{1}{\alpha}\partial^2\partial^\mu \frac{\delta}{\delta J^\mu(x)}\right) W(J,\xi,\alpha) - \partial_\mu J^\mu(x) = 0 \, (10)$$

It follows from translation and Lorenz invariance of considered model that the mean values of the fields $A^\mu, \psi$ are equal to zero. Therefore, for full propagators and full 3-point vertex it holds

$$W^{\mu\nu}_{2,0}(x_1,x_2) = \frac{\delta^2 W(J,\xi,\alpha)}{\delta J_\mu(x_1)\delta J_\nu(x)_2}\Big|_{J=\xi=0},$$

$$W_{0,2}(x_1,x_2) = \frac{\vec{\delta}}{\delta\xi(x_1)}\frac{\vec{\delta}}{\delta\xi(x_2)} W(J,\xi,\alpha)\Big|_{J=\xi=0}$$

$$W^{\mu}_{1,2}(x,x_1,x_2) = \frac{\delta}{\delta J_\mu(x)}\frac{\vec{\delta}}{\delta\xi(x_1)}\frac{\vec{\delta}}{\delta\xi(x_2)} W(J,\xi,\alpha)\Big|_{J=\xi=0}$$

# Interaction of massless MF with electromagnetic field

One obtains from (10) that

$$\partial^{\mu} W^{\mu\nu}_{2,0}(x_1, x_2) = \alpha \partial^{\mu} \partial^{-2}(x_1, x_2), \tag{11}$$

$$\partial^2 \partial_{\mu} W^{\mu}_{1,2}(x, x_1, x_2) = \alpha \kappa \bar{\gamma}^5 (\delta(x - x_2) W_{0,2}(x_1, x) - \\ - \delta(x - x_1) W_{0,2}(x, x_2)).$$

These Ward identities are analogous of ones in QED. It follows from (11) that the photonic Green functions $W_{n,0}$ for all $n > 2$ are transverse, i.e.

$$\frac{\partial}{\partial x_1^{\mu_1}} W^{\mu_1, \dots, \mu_n}_{n,0}(x_1, \dots, x_n) = 0.$$

It holds also in QED.

# Interaction of MF with material plane

Interaction of the field $\psi$ with material plane $x_3 = 0$ is described by the defect action

$$S_{def}(\psi) = \int \psi(x)\bar{\gamma}_0\Omega(x^3)\psi(x)dx,$$

with matrix $\Omega(x_3) = Q\delta(x^3)$. Since $\Omega(x_3)$ and $\delta(x_3)$ have the dimension of mass, the matrix $Q$ is dimensionless. For homogeneous isotropic material plane in more general case, the matrix $Q$ could be presented in the form:

$$Q = r_1 I + r_2\bar{\gamma}^5 + r_3\bar{\gamma}^3\bar{\gamma}^5 + r_4\bar{\gamma}^3\bar{\gamma}^5$$

with $I$ - identity 4x4 matrix, and real parameters $r_1, r_2, r_3, r_4$.

# Transmission coefficient

The characteristics of the scattering process depend essentially on the choice of parameters of the model, on polarization, energy and incidence angle of particles. For the Dirac particles moving orthogonal to the $x_3$-axes the transmission coefficient $K(k)$ has the form

$$K(k) = K_+(k)\sin(\vartheta)^2 + K_-(k)\cos(\vartheta)^2, \ \ K_\pm(k) = f_\pm(k)$$

where $k = \frac{p_0 - m}{p_0 + m}$, $p_0$ is the energy, $m$ is the mass of particle, and

$$f(k) = f(f_{max}, f_{ext}, k_{max}; k) = \frac{f_{max}}{1 + \frac{k_{max}}{f_{ext}^2}\left(\sqrt{\frac{k}{k_{max}}} - \sqrt{\frac{k_{max}}{k}}\right)^2},$$

$$f_\pm(k) = f(f_{max\pm}, f_{ext\pm}, k_{max\pm}; k).$$

Here, the parameters $f_{max}, f_{ext}, k_{max}$ are positive constants, $0 \leq f(k) \leq f_{max}$, $f_{max} = f(k_{max}) = \max f(k)$, $f_{ext} = |k_r - k_l|$, $f(k_l) = f(k_r) = f_{max}/2$.

# Transmission coefficient for Dirac particle

The function $f(q)$ can be presented as follows

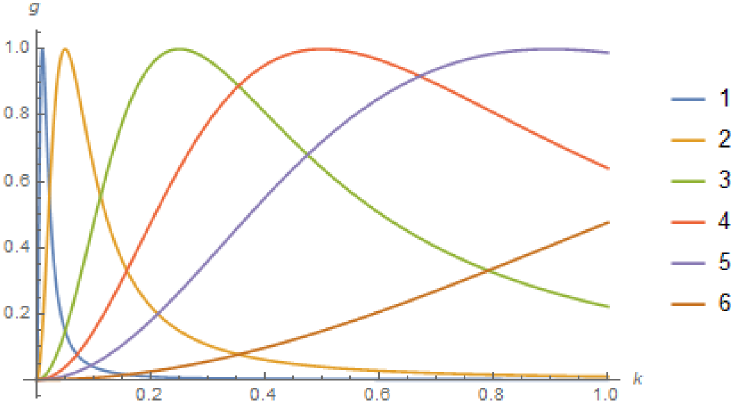
$$f(q) = \frac{f_{max} f_{ext}^2 q}{f_{ext}^2 q + (q_{max} - q)^2} = f_{max} g\left(\frac{q_{max}}{f_{ext}^2}; \sqrt{\frac{q}{q_{max}}}\right),$$

$$g(c; x) = \frac{1}{1 + c(x - x^{-1})^2}.$$

# The features of scattering processes for Dirac particle

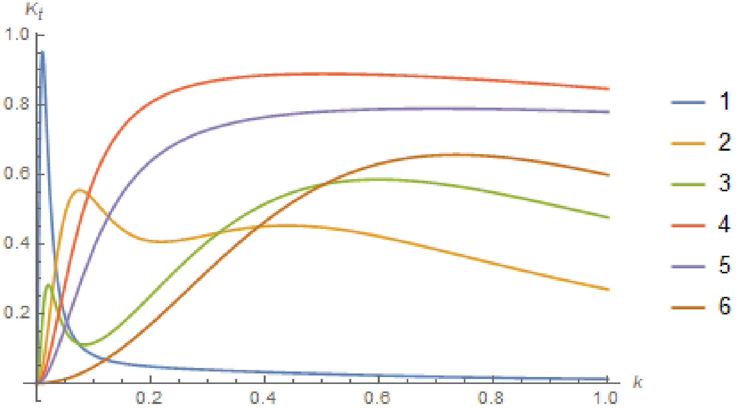
The parameters of the model can be chosen so that the transmission coefficient is almost equal to unity at low particle energy and is almost zero for particles with high energy. One can choose the parameters and so that at high energies the particles almost completely pass through the plane, and at low energies they are almost completely reflected.

# Transmission coefficient for Dirac particle




The graphs of the function $g(1/4.k/c)$ by different values of parameters $c$: 1) c=0.01; 2)c =0.05; 3)c =0.25; 4)c =0.5; 5)c =0.9; 6)c = 2.5.

# Transmission coefficient for Dirac particle




Transmission coefficients
$K_t((k)) = c_{1+}g(c_{2+}, k/c_(3+))\cos(\vartheta)^2 + c_{1-}g(c_{2-}, k/c_{3-})\sin(\vartheta)^2$
by different values of $c_{1\pm}$, $c_{1\pm}$, $c_{1\pm}$, $\vartheta$.

# Transmission coefficient for Dirac particle

The parameters $c_{1\pm}, c_{2\pm}, c_{3\pm}, \vartheta$.

1) $c_{1+} = 0.99, c_{2+} = 0.225, c_{3+} = 0.01, c_{1-} = 0.8, c_{2-} = 0.025, c_{3-} = 0.1, \cos(\vartheta)^2 = 0.95$;

2) $c_{1+} = 0.95, c_{2+} = 0.225, c_{3+} = 0.07, c_{1-} = 0.9, c_{2-} = 0.25, c_{3-} = 0.5, \cos(\vartheta)^2 = 0.55$;

3)$c_{1+} = 0.8, c_{2+} = 0.25, c_{3+} = 0.02, c_{1-} = 0.9, c_{2-} = 0.2, c_{3-} = 0.6, \cos(\vartheta)^2 = 0.35$;

4) $c_{1+} = 0.9, c_{2+} = 0.025, c_{3+} = 0.5, c_{1-} = 0.8, c_{2-} = 0.0025, c_{3-} = 0.7, \cos(\vartheta)^2 = 0.9$;

5) $c_{1+} = 0.8, c_{2+} = 0.025, c_{3+} = 0.7, c_{1-} = 0.7, c_{2-} = 0.0025, c_{3-} = 0.9, \cos(\vartheta)^2 = 0.9$;

6) $c_{1+} = 0.6, c_{2+} = 0.25, c_{3+} = 0.7, c_{1-} = 0.8, c_{2-} = 0.25, c_{3-} = 0.8, \cos(\vartheta)^2 = 0.7$.

# Conclusions

An analysis of the possibility of generalizations of the Majorana theory based on the assumption of the realness of this spinor field has been carried out using the general principles of relativistic physics. For it the action functional is constructed, which contains two parameters: one has dimension of mass the second is dimensionless. It is found the complete solutions of the Euler-Lagrange equations. An explicit form of the eigenspinors of the helicity operator is obtained, which can be used as basis elements in describing the states of the system. In it, in agreement with the results obtained by Majorana, the ordinary electric current is zero. The axial current is nontrivial. Its basic features have been studied and, within the framework of the constructed modification of Majorana theory, the possibility of participation of the real spinor fermion field in electromagnetic interactions is demonstrated.

# Thank you for your attention!