Exact and approximate dualities between various phenomena in QCD phase diagram

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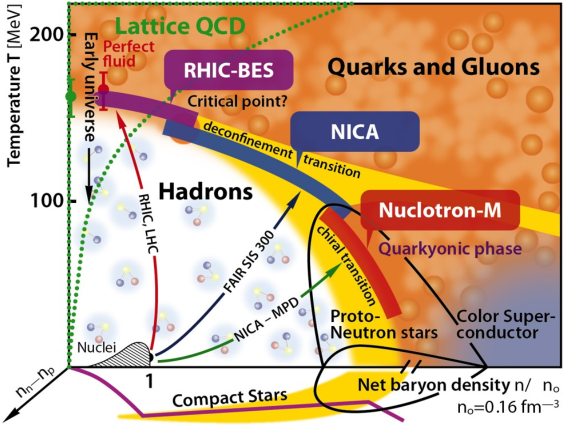
QCD Phase Diagram

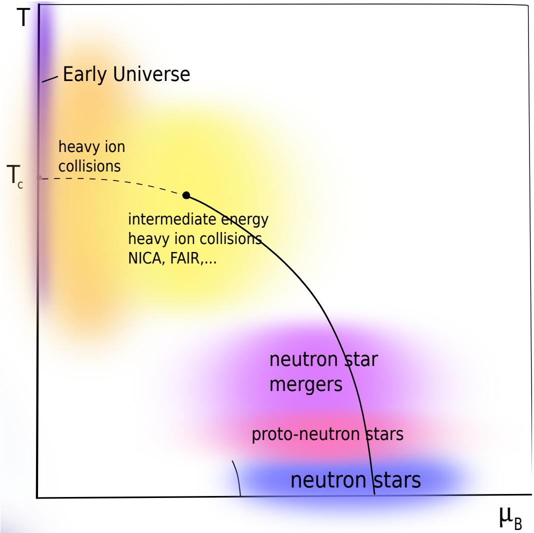


Two main phase transitions

- confinement-deconfinement
- chiral symmetry breaking phase—chriral symmetric phase

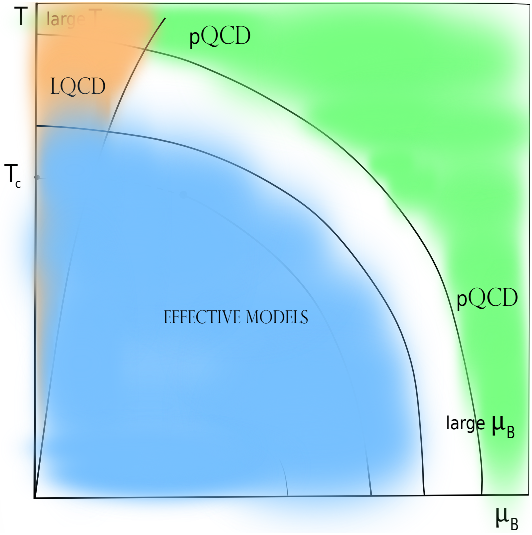
QCD at T and μ
(QCD at extreme conditions)

- Early Universe
- heavy ion collisions
- neutron stars
- proto- neutron stars
- neutron star mergers



Methods of dealing with QCD

- Perturbative QCD
- First principle calculation – lattice QCD
- Effective models
- DSE, FRG
- Gauge/Gravity duality
- .....



Isospin imbalance

- **Isotopic chemical potential μ_I**

  Allow to consider systems with isospin imbalance ($n_n \neq n_p$).

- Neutron stars, intermediate energy heavy-ion collisions, neutron star mergers

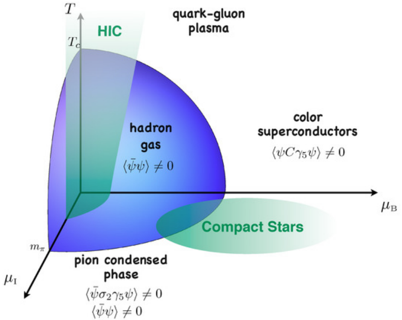


Figure: *taken from Massimo Mannarelli*

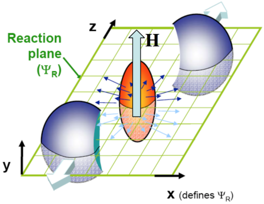
$$\frac{\mu_I}{2}\bar{q}\gamma^0\tau_3 q = \nu\left(\bar{q}\gamma^0\tau_3 q\right) \qquad n_I = n_u - n_d \quad \longleftrightarrow \quad \mu_I = \mu_u - \mu_d$$

- **Chiral (axial) chemical potential**

Allow to consider systems with chiral imbalance.

$$n_5 = n_R - n_L, \quad \mu_5 = \mu_R - \mu_L, \ \vec{J} \sim \mu_5 \vec{B},$$

The corresponding term in the Lagrangian is $\mu_5 \bar{q}\gamma^0\gamma^5 q$



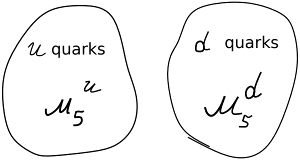
- **Chiral isospin chemical potential**

$$\mu_5^u \neq \mu_5^d \text{ and } \mu_{I5} = \mu_5^u - \mu_5^d$$

Term in the Lagrangian

$$-\quad \frac{\mu_{I5}}{2}\bar{q}\tau_3\gamma^0\gamma^5 q = \nu_5(\bar{q}\tau_3\gamma^0\gamma^5 q)$$

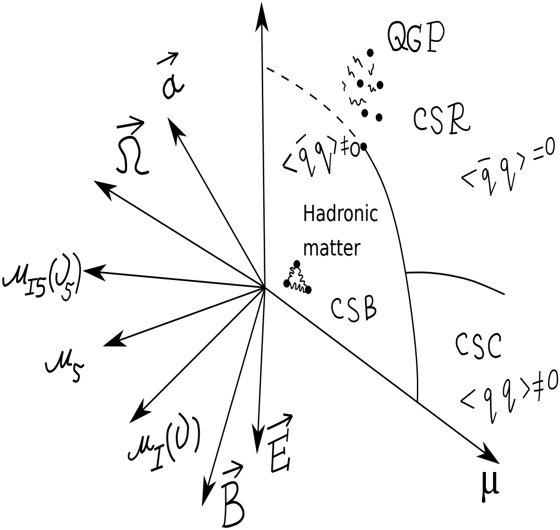
$$n_{I5} = n_{u5} - n_{d5}, \qquad n_{I5} \quad \longleftrightarrow \quad \nu_5$$



More external conditions to QCD

More than just QCD at (μ, T)

- more chemical potentials μ_i
- magnetic fields
- rotation of the system $\vec{\Omega}$
- acceleration $\vec{a}$
- finite size effects (finite volume and boundary conditions)



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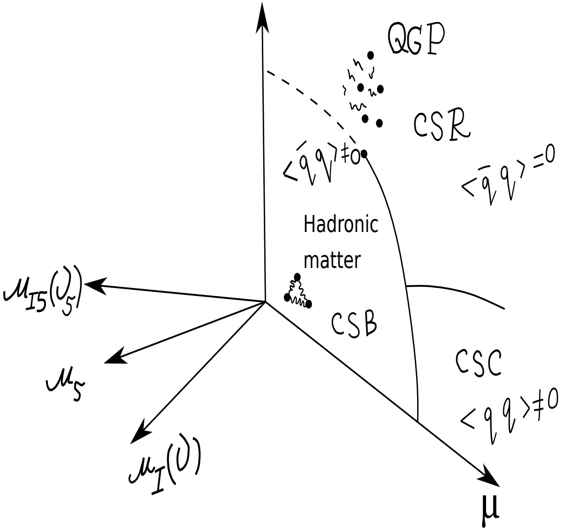

Recall that in effective models there have been found **dualities**

(*It is not related to holography or gauge/gravity duality*)

Chiral symmetry breaking $\Longleftrightarrow$ pion condensation

Isospin imbalance $\Longleftrightarrow$ Chiral imbalance

The TDP

$$\Omega(T, \mu, \mu_i, ..., \langle \bar{q}q \rangle, ...) \qquad \Omega(T, \mu, \nu, \nu_5, ..., M, \pi, ...)$$

$M = \langle \sigma(x) \rangle \sim \langle \bar{q}q \rangle,$ CSB phase: $M \neq 0,$

$\pi_1 = \langle \pi_1(x) \rangle = \langle \bar{q}\gamma^5 \tau_1 q \rangle,$ PC phase: $\pi_1 \neq 0,$

The TDP

$$\Omega(T, \mu, \mu_i, ..., \langle\bar{q}q\rangle, ...) \qquad \Omega(T, \mu, \nu, \nu_5, ..., M, \pi, ...)$$

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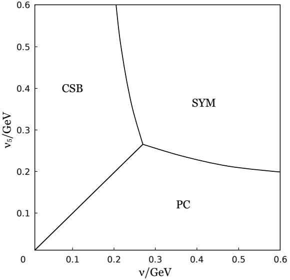


$$\mathcal{D}: \; M \longleftrightarrow \pi, \quad \nu \longleftrightarrow \nu_5$$

Duality between chiral symmetry breaking and pion condensation

$$\text{PC} \longleftrightarrow \text{CSB} \quad \nu \longleftrightarrow \nu_5$$

Dualities in QC_2D

- similar phase transitions:

  confinement/deconfinement, chiral symmetry breaking/restoration
- A lot of physical quantities coincide with some accuracy

  Critical temperature, shear viscosity etc.
- There is **no sign problem** in SU(2) case and lattice simulations at non-zero baryon density are possible — $(Det(D(\mu)))^{\dagger} = Det(D(\mu))$

It is a great playground for studying dense matter

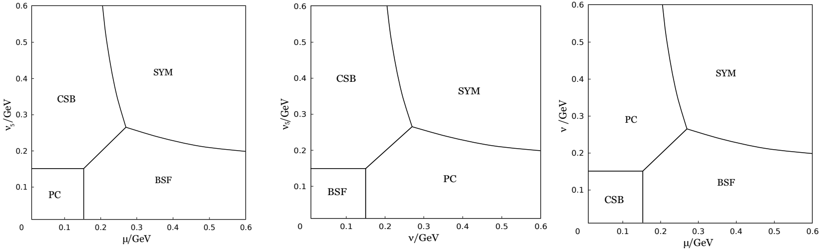
Condensates and phases

$M = \langle \sigma(x) \rangle \sim \langle \bar{q}q \rangle,$ CSB phase: $M \neq 0,$

$\pi_1 = \langle \pi_1(x) \rangle = \langle \bar{q}\gamma^5 \tau_1 q \rangle,$ PC phase: $\pi_1 \neq 0,$

$\Delta = \langle \Delta(x) \rangle = \langle qq \rangle = \langle q^T C \gamma^5 \sigma_2 \tau_2 q \rangle,$ BSF phase: $\Delta \neq 0.$

Dualities in QC_2D

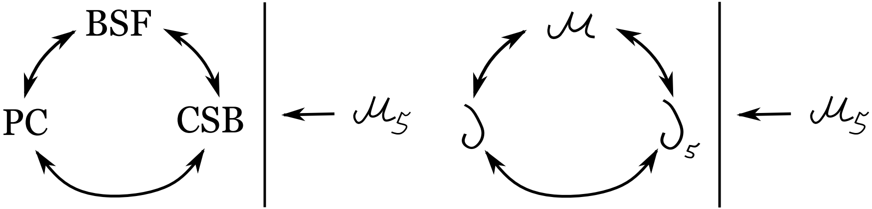


$$(a) \qquad \mathcal{D}_1: \quad \mu \longleftrightarrow \nu, \quad \pi_1 \longleftrightarrow |\Delta|, \qquad \text{PC} \longleftrightarrow \text{BSF}$$

J. Andersen, T. Brauner, D. T. Son, M. Stephanov, J. Kogut, ...

$$(b) \qquad \mathcal{D}_3: \quad \nu \longleftrightarrow \nu_5, \quad M \longleftrightarrow \pi_1, \qquad \text{PC} \longleftrightarrow \text{CSB}$$

$$(c) \qquad \mathcal{D}_2: \quad \mu \longleftrightarrow \nu_5, \quad M \longleftrightarrow |\Delta|, \qquad \text{CSB} \longleftrightarrow \text{BSF}$$



μ, ν and ν_5 are connected by dualities.

Chiral imbalance μ_5 does not participate in dual transformations

Uses of Dualities

How (if at all) it can be used

discussed in Particles 2020, 3(1), 62-79

$$(\mu, \nu) \longrightarrow (\mu, \nu_5)$$

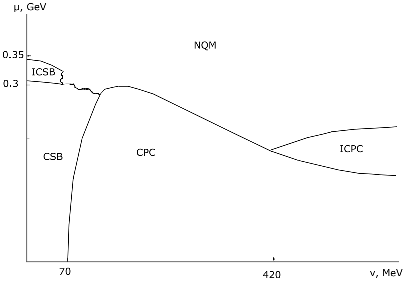


Figure: (ν, μ)-phase diagram.

M. Buballa, S. Carignano, J. Wambach, D. Nowakovski, Lianyi He et al.

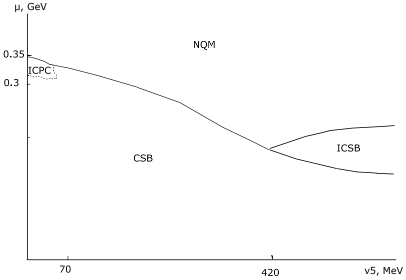
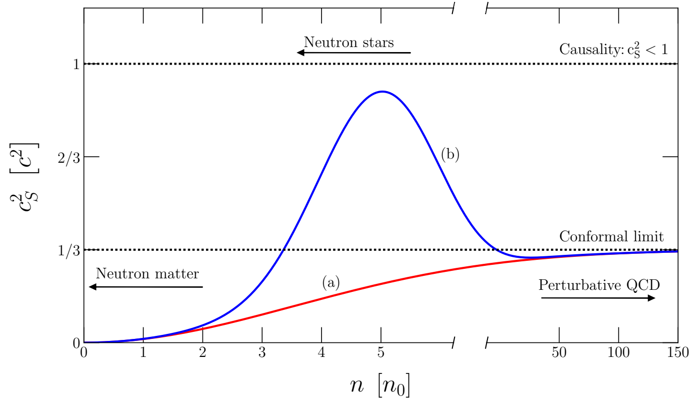


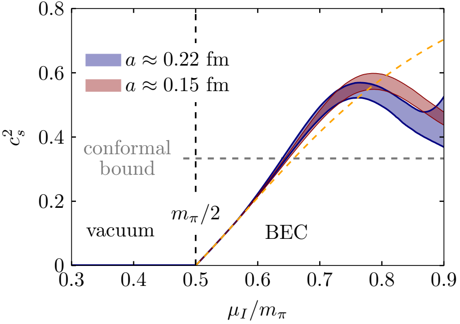
Figure: (ν_5, μ)-phase diagram

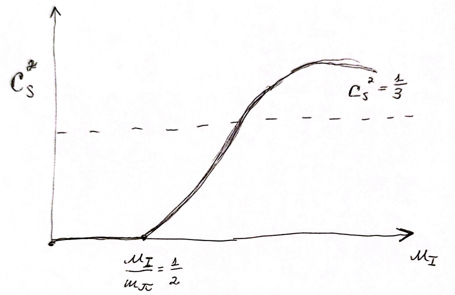
Two possible scenario of speed of sound at non-zero baryon density



taken from S. Reddy et al, Astrophys. J. ***860*** *(2018) no.2, 149*

Sound speed in QCD at μ_I: skematic



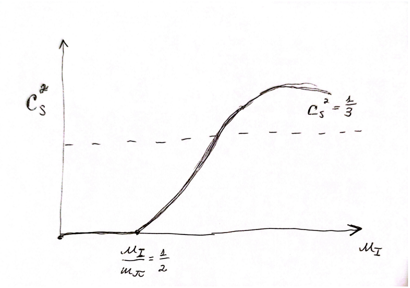


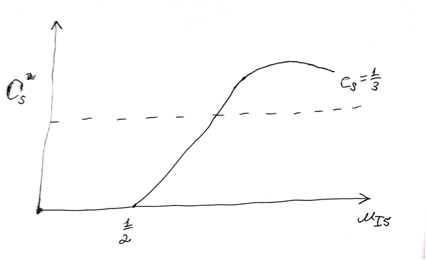
Sound speed squared has been obtained from **lattice QCD** for **QCD with non-zero isospin** μ_I

B. B. Brandt, F. Cuteri and G. Endrodi, JHEP 07, 055 (2023)

R. Abbott et al. [NPLQCD], Phys. Rev. D 108, no.11, 114506 (2023)

Speed of sound in QCD: First principles



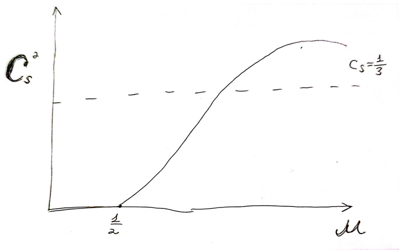


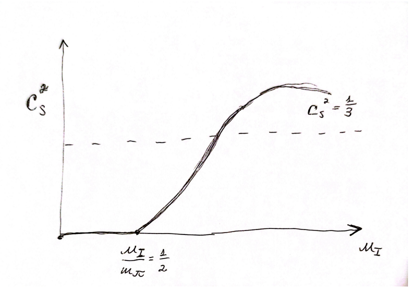
Speed of sound

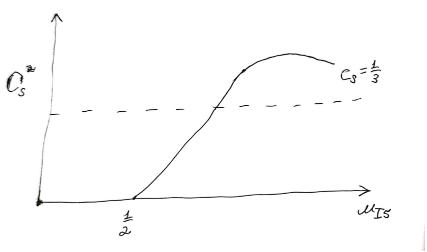
in two colour QCD case

QC_2D

Speed of sound in QC$_2$D: First principle







- A lot of densities and imbalances
  baryon, isospin, chiral, chiral isospin imbalances
- Finite temperature $T \neq 0$
- Physical pion mass $m_\pi \approx 140$ MeV
- Inhomogeneous phases (case)

$$\langle \sigma(x) \rangle = M(x), \quad \langle \pi_\pm(x) \rangle = \pi(x), \quad \langle \pi_3(x) \rangle = 0.$$

- Inclusion of color superconductivity phenomenon

Lagrangian of two colour QCD can be written in the form

$$\mathcal{L} = i\bar{\Psi}\gamma^{\mu}D_{\mu}\Psi$$

where $D_{\mu} = \partial_{\mu} + igA_{\mu} = \partial_{\mu} + ie\sigma_a A_{\mu}^a$

$$\Psi^T = \left(\ \psi_L^u,\ \ \psi_L^d,\ \ \sigma_2(\psi_R^C)^u,\ \ \sigma_2(\psi_R^C)^d\ \right)$$

Flavour symmetry is $SU(4)$

Pauli-Gursoy symmetry

Dualities $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ were found in were found in

- In the framework of NJL model beyond mean field
- From first principles. In $\mathrm{QC_2D}$ at the level of Lagrangian

Duality was shown in

- In the framework of NJL model beyond mean field or without any approximation
- From first principles

Phase diagram of QCD
and
color superconductivity
at non-zero chiral imbalance

There are only three order parameters

$$M = \langle\sigma(x)\rangle = -2G\langle\bar{q}q\rangle, \quad \pi = \langle\pi_1(x)\rangle = -2G\langle\bar{q}i\gamma^5\tau_1 q\rangle,$$

$$\Delta = \langle\Delta(x)\rangle = -2H\langle\overline{q^c}i\gamma^5\tau_2\lambda_2 q\rangle$$

The equations of motion for bosonic fields

$$\sigma(x) = -2G(\bar{q}q), \quad \pi_a(x) = -2G(\bar{q}i\gamma^5\tau_a q),$$

$$\Delta_A(x) = -2H(\overline{q^c}i\gamma^5\tau_2\lambda_A q), \quad \Delta^*_A(x) = -2H(\bar{q}i\gamma^5\tau_2\lambda_A q^c)$$

the Lagrangian and the effective action are invariant under the color $SU(3)_c$ group, hence the TDP depends on the combination

$$\Delta_2\Delta^*_2 + \Delta_5\Delta^*_5 + \Delta_7\Delta^*_7 \equiv \Delta^2, \quad \text{where } \Delta \text{ is a real quantity.}$$

Condensates and phases

$M = \langle \sigma(x) \rangle \sim \langle \bar{q}q \rangle \neq 0,$ CSB phase:

$\pi = \langle \pi_1(x) \rangle = \langle \bar{q}\gamma^5 \tau_1 q \rangle \neq 0,$ PC phase: $\pi_1 \neq 0$

$\Delta = \langle \Delta(x) \rangle = \langle \overline{q^c} i \gamma^5 \tau_2 \lambda_2 q \rangle = \langle qq \rangle \neq 0,$ CSC phase: $\Delta \neq 0$

$$m_\pi,\ f_\pi,\ \langle \bar{q}q \rangle \longrightarrow \text{quark-antiquark coupling } G$$

H is not precisely determined

If the quark-antiquark interaction has been constrained empirically, the most natural solution is to determine the quark-quark coupling constants empirically, too. Unfortunately, the analog to the meson spectrum would be a diquark spectrum, which of course does not exist in nature

The most natural fit is

$$H = \frac{3}{4}G = 0.75G$$

- from Fiertz transform
- or from reasonable value of condensate

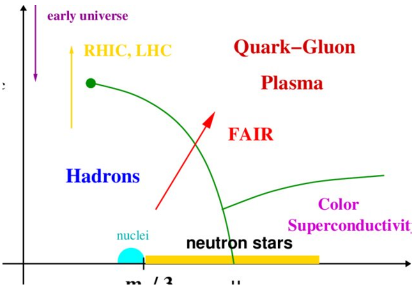
But we can use $0 < H < G$

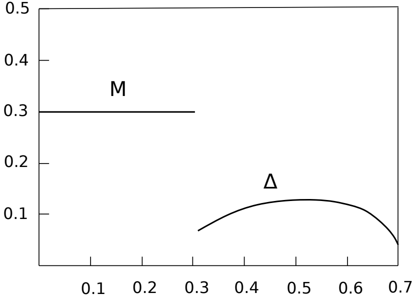
If we one consider unphysical twice as strong diquark channel

$$H = \frac{3}{2}G = 1.5\,G$$

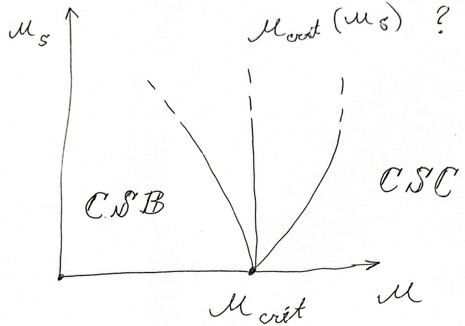
It will be very instructive later

CSC phenomenon and diqurak condensate



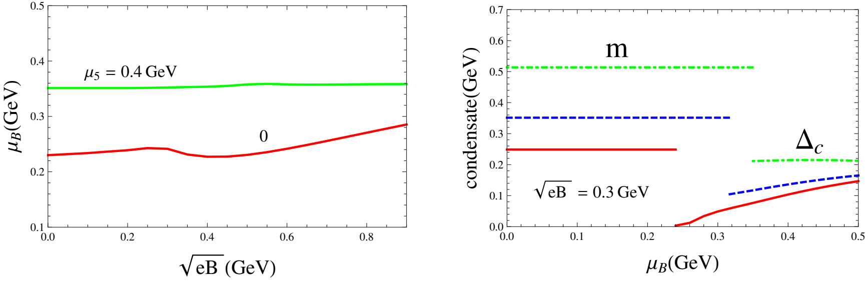


Color superconductivity at non-zero μ_5



How chiral imbalance μ_5 influence the generation of color superconductivity

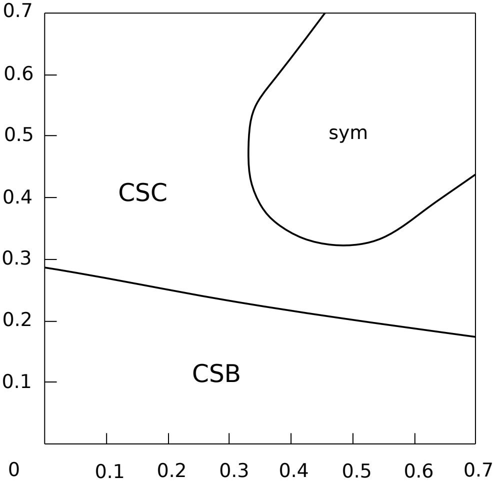
Phase structure: (μ_5, μ)-phase diagram



Chiral imbalance μ_5 inhibits the generation of color superconductivity

*G. Cao and P. Zhuang, Phys. Rev. D **92** (2015) no.10, 105030*

Phase structure: (μ_5, μ)-phase diagram

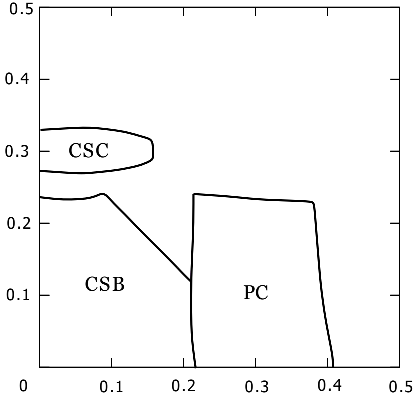


Chiral imbalance μ_5 *facilitates the generation of color superconductivity*

Two regularization schemes have been used but further clarification is required

Diquark condensation at $\mu = 0$ in SU(3)

Chiral imbalance μ_5 leads to the **diquark condensation** in the region of the phase diagram at $\mu = 0$ in three color case

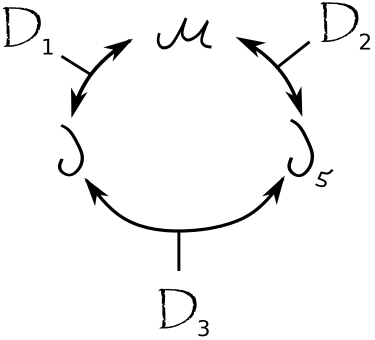


Approximate dual properties

with color superconductivity phenomenon

in three color case

Duality structure in QC_2D



D_1: $\mu \longleftrightarrow \nu$

D_2: $\mu \longleftrightarrow \nu_5$

D_3: $\nu \longleftrightarrow \nu_5$

Duality structure in QCD

$$\Omega(M, \pi, \Delta, \mu, \nu, \nu_5, \mu_5)$$

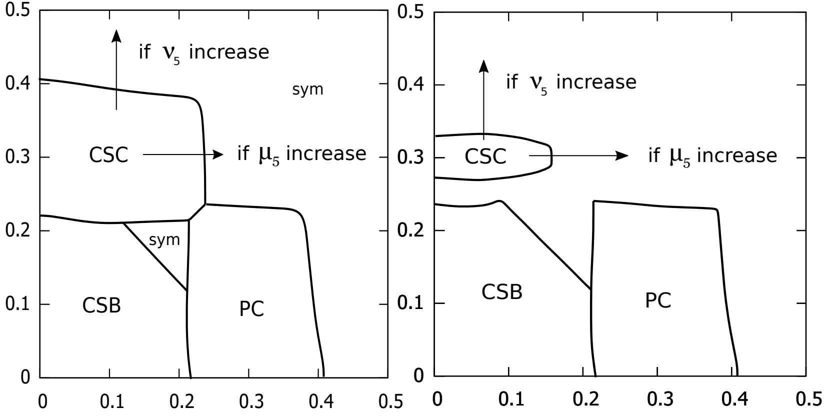
$D_1: \quad \mu \longleftrightarrow \nu$

$D_2: \quad \mu \longleftrightarrow \nu_5$

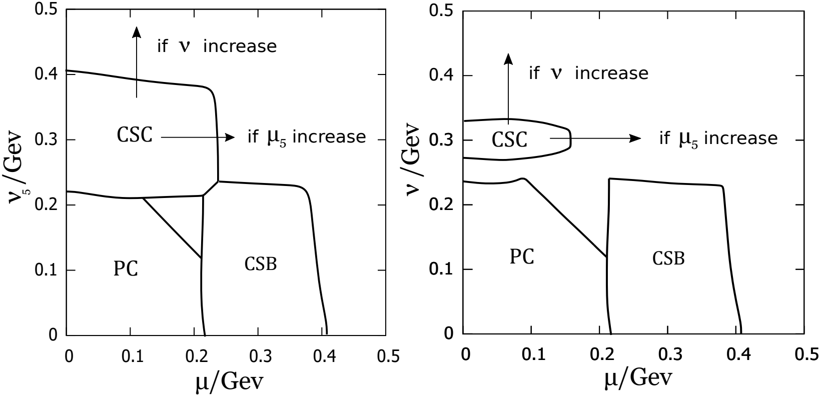
$D_3: \quad \nu \longleftrightarrow \nu_5$

QCD

Approximate or qualitative dual properties



$(\mu,\, \nu)$-phase diagram at $\nu_5 = 0.3$ GeV and $\mu_5 = -0.5$. Hard cut-off. Left: $H = 1.5\, G$. Right: $H = 0.75\, G$.



$(\mu,\, \nu_5)$-phase diagram at $\nu = 0.3$ GeV and $\mu_5 = -0.5$. Hard cut-off. Left: $H = 1.5\, G$. Right: $H = 0.75\, G$.

Unphysically strong quark-quark interactions

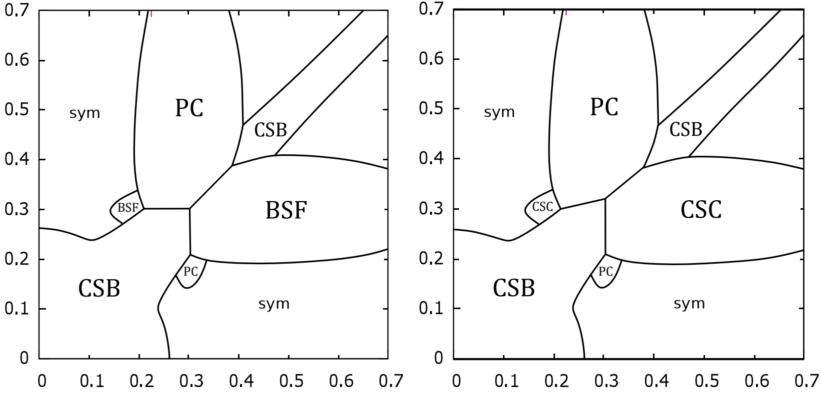


Figure: $(\mu,\, \nu)$-phase diagram at $\nu_5 = 0.3$ GeV and $\mu_5 = 0$.
Left: two color QCD. Right: Three color QCD at $H = 1.5\, G$.
The Phase diagram is drawn in the framework of NJL models

As a rule (not necessarily) **there is no mixed phase** with $M \neq 0$, $\pi \neq 0$ and $\Delta \neq 0$

The projections of TDP are

$$F_1\left(M, \mu_i\right) = \Omega(M, \pi, \Delta, \mu_i)\Big|_{\pi=0,\ \Delta=0}$$

$$F_2\left(\pi, \mu_i\right) = \Omega(M, \pi, \Delta, \mu_i)\Big|_{M=0,\ \Delta=0}$$

$$F_3\left(\Delta, \mu_i\right) = \Omega(M, \pi, \Delta, \mu_i)\Big|_{M=0,\ \pi=0}$$

The TDP projections have the following structure

$$F_1\,(M,\mu_i) = f(M,\mu_i)$$

$$F_2\,(\pi,\mu_i) = \mathcal{D}_3 f(\pi,\mu_i) = \mathcal{D}_1 F_3(\pi,\mu_i) + \bar{g}(T,\mu_i)$$

$$F_3\,(\Delta,\mu_i) = \mathcal{D}_2 F_1\,(\Delta,\mu_i) + \bar{f}(\mu_i)$$

$$= \mathcal{D}_1\big(\mathcal{D}_3 f(M,\mu_i)\big) + \bar{f}_1(T,\mu_i)$$

Gap equations are dual with respect to each other so the condensates

$$\frac{\partial F_1\left(M,\mu_i\right)}{\partial M}=0,\qquad \frac{\partial F_2\left(\pi,\mu_i\right)}{\partial \pi}=0,\qquad \frac{\partial F_3\left(\Delta,\mu_i\right)}{\partial \Delta}=0$$

$D_1:\ \mu \longleftrightarrow \nu$

$D_2:\ \mu \longleftrightarrow \nu_5$

$D_3:\ \nu \longleftrightarrow \nu_5$

Unphysically strong quark-quark interactions

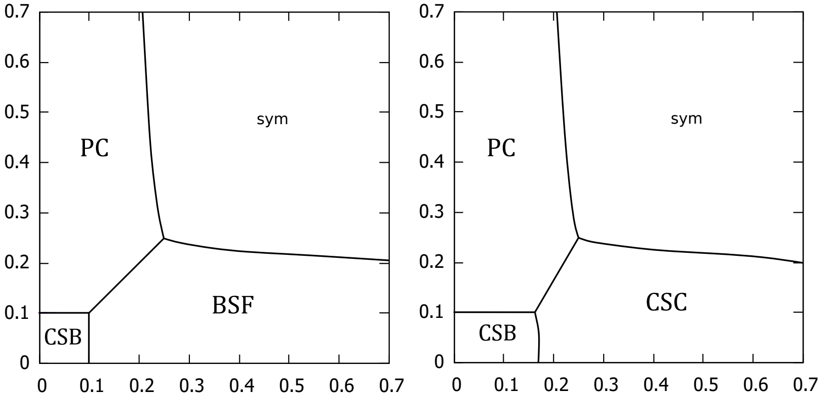


Figure: (μ, ν)-phase diagram at $\nu_5 = 0.1$ GeV and $\mu_5 = 0$.
Left: two color QCD. Right: Three color QCD at $H = 1.5\,G$.
The phase diagram is drawn in the framework of NJL models

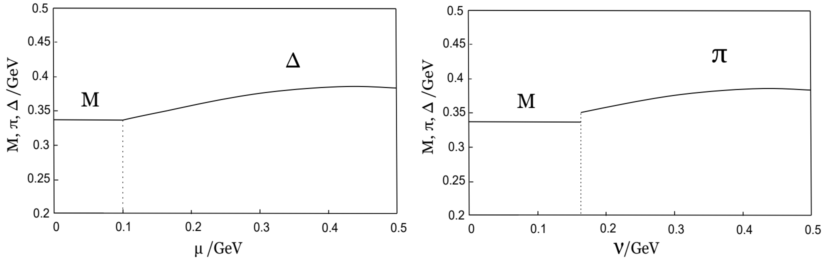
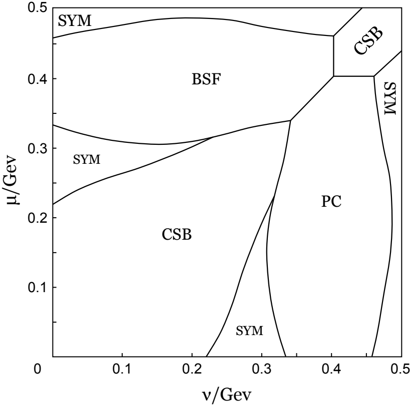
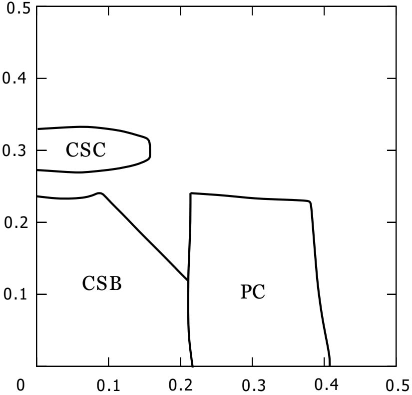


Figure: Values of condensates as a function of Left: μ at $\nu_5 = 0.1$ GeV, $\nu = 0.05$ GeV and $\mu_5 = 0$. Right: ν at $\nu_5 = 0.1$ GeV, $\mu = 0.05$ GeV and $\mu_5 = 0$.

Diquark condensation at $\mu = 0$: SU(2) and SU(3)





Chiral imbalance μ_5 leads to the **diquark condensation** at $\mu = 0$.

It was first shown in **two color case** and then for **three color**.

Conclusions and Thanks for the Attention

- It was shown that there exist dualities in QCD and QC_2D

  Richer structure of Dualities in the two colour case

- There have been shown ideas how dualities can be used

  Duality is not just entertaining mathematical property but an instrument with very high predictivity power

- Dualities have been shown from first principles in the two colour and three color QCD

- Approximate dualities have been discovered in QCD in framework of eff models